Discussion Papers In Economics And Business

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Abstract

Why do well-established companies often lose managerial fidelity due to corporate scandals, and how do they restructure management to recover from corporate failures? In this study, we present a dynamic theory of management cycles by which firms endogenously switch between different management regimes over time. Firms accumulate managerial capital as intangible assets such as managerial knowledge, know-how, and skills over time. We show that current managers of a firm are disciplined by not only how much managerial capital accumulated through the prior business operations within the firm, but also by the market valuation about future profitability of the firm. Through such dynamic interactions, we show that management cycles endogenously emerge and persist over time.

Keywords: Dynamic Enforcement, Managerial Capital, Moral Hazard

JEL Classification Numbers: D21, D86, M21

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1 Introduction

1.1 Motivation and Results

In this study, we present a dynamic theory of management cycles to account for how and why firms switch between different management regimes with fluctuating profits over time.

Corporate management is not stationary, but rather changes over time as profits rise and fall. Why do many well-established companies often lose managerial fidelity through corporate scandals, such as concealing product defects, financial fraud, and window dressing? Management failures caused many world-renowned companies to lose their reputations in the market and suffered significant drops in profitability (see Gray, Frieder, and Clark, Jr (2005) for many cases of corporate scandals worldwide). More interestingly, although scandals forced some companies into bankruptcy, others successfully recovered profitability. For example, the diesel emissions scandal in 2015 caused Volkswagen to lose profits,[1] though it managed to increase profits in just a few years after the scandal.[2] Barclays also experienced a drop in profits in 2015 because it had to provide a large fund to pay for the fines and litigation costs related to the scandal surrounding the manipulation of foreign currency markets in 2012.[3] However, Barclays recovered its profitability soon after the scandal. In fact, its profits nearly trebled in 2016.[4]

Despite the many anecdotes of corporate successes and failures, the literature contains few theoretical attempts to investigate the dynamic nature of management practices, such as how and why firms switch between better and worse management as profits fluctuate over time. Our study aims to address this gap. Specifically, we develop a dynamic model to show that firms do not sustain a particular management practice, but rather alternate between different management regimes over time.

We also show that such "swings" in management are associated with changes in the firm's performance over time. One relevant empirical finding indicates that firms' management practices play the important role of determining their productivities and performances (Bloom and Van-Reenen 2007; 2010; Bloom, Sadun, and Van-Reenen 2015; 2016). On this point, several papers identify the significant effects of intangible assets accumulated within firms, such as knowledge, know-how, corporate cultures, and customer relationships on firms' profitability (Hall 2001; Brynjolfsson and Hitt 2003; Lev and Radhakrishnan 2003; Peters

[1]Volkswagen sold cars with emissions higher than that reported in diesel emissions laboratory tests (*The Economist* (Nov 7, 2015)).

[2]In the first nine months of 2017, the profits of the Volkswagen brand doubled to 2.5 billion Euro (*Financial Times* (Oct, 2017, Jan 18, 2018)).

[3]*The New York Times* (April 30, 2015).

[4]In 2016, Barclays's profits trebled to 3 billion GBP from 1 billion GBP in 2015 (*The Daily Telegraph* (Feb 24, 2017)).

and Taylor 2017).[5] Regarding the changes in corporate profits, Hu and Johri (2018) find that corporate profits in the US are much more volatile than real GDP is. We illustrate a related fact in Figure 1 showing that real corporate profits change more rapidly over time than real GDP per capita does in the US. Interestingly, Hu and Johri (2018) show that a firm's intangible assets may be a relevant factor to explain such high volatility in profits. We show that the endogenous emergence of management cycles results in large fluctuations of aggregate profits relative to real GDP, which is consistent with this finding.

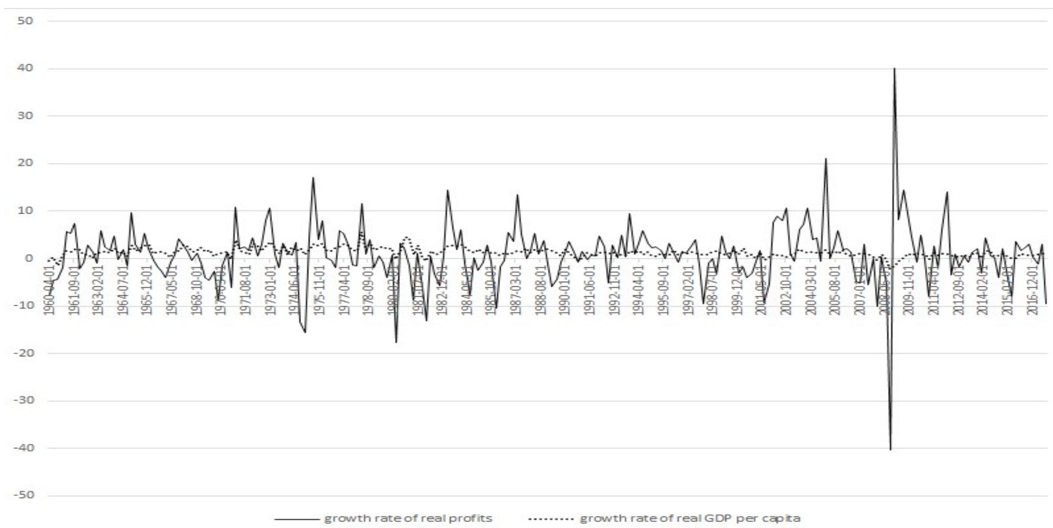

Figure 1. Profits and GDP in the US.[6]

In this paper, we refer to a firm's accumulated intangible assets as *managerial capital*, which successive generations of managers within firms inherit over time. The novel feature of our model is that we investigate the dynamic interactions between the accumulation of managerial capital and the limited commitment to contracts, which results in endogenous switching between different management regimes over time. As firms accumulate more managerial capital, they can exploit higher productivities and hence produce more in the current period. However, managerial capital accumulation is limited by the *dynamic enforcement constraint* (DE constraint), meaning that firms need to secure payments to trading parties in

[5]These intangible assets are often called "organizational capital" in the literature. For example, see Corrado, Haltiwanger, and Sichel (2005), Lev (2003), and Eisfeldt and Papanikolaou (2013) for the issues around measuring organizational capital.

[6]The dashed lines indicate the quarterly growth rate of real corporate profits before tax in the US for 1960-2017. The straight lines indicate the quarterly growth rate of real GDP per capita in the US during the same period. These are computed by using 2009 as the base year. Data source: the Economic Research of the Federal Reserve of St. Louis (https://fred.stlouisfed.org/).

a self-enforcing manner because their business activities and revenues are not fully verifiable. The key to our model is that each firm needs to commit to paying multiple parties jointly. These parties provide different production inputs simultaneously.

In our model economy firms produce outputs by using managerial capital and other production inputs, called "widget". Managerial capital accumulates within firms by motivating the hired managers to acquire management qualities (exert managerial efforts) while widget producers produce widgets. All management quality and widget production are non-verifiable, and are hence not formally contractible. Thus, firms cannot commit to formal contracts to pay managers and widget producers. On the other hand, firms make informal agreements with managers and widget producers about management qualities and widget productions. Such informal agreements are self-enforced as long as firms lose their future market reputations by deviating from these agreements. Specifically, firms suffer from lower market value; that is, their stock prices, when they lose market reputation. Thus, firms can commit to paying managers and widget producers only up to their market values, which are determined in the market based on expected future profits. This firm value then determines *the commitment capacity* for a firm to self-enforce informal agreements.

The important consequence of the DE constraint is that, for managers to be motivated to acquire higher management quality, firms need to offer higher incentive pay, which must however reduce payments to widget producers because the firm has a limited commitment capacity. Then, as a firm accumulates enough managerial capital, the firm finds it less profitable to increase it further by having the current managers work harder to enhance management quality. Rather, the firm switches to reduce managers' incentive pay, which can relax the commitment capacity. By utilizing the expanded commitment capacity, firms can procure more widgets from widget producers and pay them more. This strategy becomes more profitable for the firm as managerial capital accumulates more because the marginal product of managerial capital decreases more than that of widget does due to diminishing marginal returns. Then, firms fail to sustain productive management practices and then tend to lower managerial productivity when managerial capital accumulation exceeds some critical point.

For the opposite reason, when managerial capital becomes so low that its marginal product is high, firms have more incentives to increase managerial capital by motivating managers to acquire higher management quality while using less production inputs from other trading parties. Thus, a firm's managerial capital tends to expand once it reaches a low cutoff point, recovering the firm's profitability.

According to the dynamic process described above, we show that every equilibrium must be *cyclical* in the sense that firms endogenously switch between management quality over time, provided that firms can accumulate managerial capital at a sufficiently high rate and that the lowest management quality contributes little to managerial capital accumulation.

On the one hand, if an equilibrium exists that converges to the implementation of a management quality above the lowest level, then managerial capital eventually becomes large enough, provided the firm can accumulate it at a rapid pace. However, as we discussed above, firms then tend to have managers who choose a lower management quality once managerial capital reaches a sufficiently high level. On the other hand, if an equilibrium exists that converges to the implementation of the lowest management quality, managerial capital eventually tends to be low, provided that the lowest management quality contributes little to managerial capital accumulation. Since a lower managerial capital has its larger marginal product than that of widget, firms switch to induce managers to work hard to acquire higher management quality and increase managerial capital at the expense of reduced widget production. In this way, we show that managerial capital does not monotonically change but moves cyclically between high and low levels over time in every equilibrium path.

To obtain these endogenous management cycles, we emphasize the role of the DE constraint, without which the equilibrium path never exhibits cyclical features. If firms were not constrained by their commitment capacity at all, their managerial capital would monotonically converge to a unique steady state. On the contrary, if the DE constraint is binding, managerial capital is augmented only by having managers work hard, but this reduces the other production inputs. Whether this substitution effect favors the enhancement of managerial capital or other production inputs depends on the amount of managerial capital accumulated within the firm in the past periods. We show that this effect is essential in the dynamic changes of managerial capital over time.

1.2 Related Literature

We base our model on an overlapping generations economy in which three types of individuals, called entrepreneurs, managers, and widget producers, are born every period and live for two periods. Young entrepreneurs purchase the ownership of a firm from old entrepreneurs at market prices, and they then run the firm by hiring managers and widget producers. The market value of a firm, its stock price in this case, limits the extent of the entrepreneur's commitment to pay hired managers and widget producers. In particular, when old entrepreneurs expect to sell their ownership at a higher market price, they can commit to paying hired managers and widget producers at higher levels. This determines how firms self-enforce agreed upon contracts with managers and widget producers.

Some studies address a similar issue by focusing on the role of a firm's reputation as a traded asset that sustains self-enforcing agreements, even among finitely-lived players (Kreps 1990; Tadelis (1999; 2002): the firm's "names" are sold at higher prices to the next generation when the current owners operate the firms honestly than when they behave dishonestly. This gives firm owners the incentive to maintain good reputations, even when they live for finite periods, which is in line with our research interest. However, our study differs in that we investigate the dynamic interactions between the market values of firms that sustain self-enforcing agreements and the accumulation of an endogenous state variable, which is management capital in our study, over time. We show that managerial capital accumulation affects and is affected by the extent of how much entrepreneurs can commit to honoring agreed upon payments to trading parties, which prior studies did not address.

Furthermore, a new aspect of our model is that entrepreneurs need to commit to paying multiple trading parties (managers and widget producers) at the same time. Due to the DE constraint, entrepreneurs must pay widget producers less, who then produce fewer widgets when they want hired managers to work harder and pay more. In this way, managers' efforts and widget producers' inputs become substitutes through the DE constraint. Barron, Li, and Zator (2018) consider a related model in which a principal pays a loan to a creditor while motivating an agent to work hard over time. However, we focus on the interaction between the DE constraint and managerial capital in determining management cycles, which they do not consider in their study.

Furthermore, there is a large body of macroeconomics literature on endogenous credit constraints with financial frictions (Brunnermeier, Eisenbach, and Sannikov 2012) and the dynamic theory of financing (Albuquerque and Hopenhayn 2004; Clementi and Hopenhayn 2006). Among others, some studies consider the role of asset prices in determining how much firms can commit to repaying creditors (Miao and Wang 2018; Martin and Ventura 2017). However, the main focus of these studies is the emergence of rational bubbles and their effects on economic growth, which differs from our focus.

The remaining sections are organized as follows. In section 2 we set up the basic model. In section 3, we derive the optimal contract design for entrepreneurs, managers, and widget producers, given the managerial capital accumulated in the past period and the market value of the firm. In section 4, we then consider the dynamic processes of how the market values of firms and the evolution of managerial capital change over time jointly. Next, we show that every equilibrium must be cyclical and that a cyclical equilibrium exists. Section 5 provides the results for the volatility of equilibrium profits. Section 6 includes some extensions of the model. We provide all proofs in Appendix A, and some extensions in Appendix B and C.

2 Model

2.1 Environment

We consider an overlapping generations economy with a single good. Time is discrete, indexed by t, and extended over infinity, $t = 0, 1, 2, ...$ In each period, three types of individuals, called *entrepreneurs, widget producers, and managers*, are born and each lives for two periods. There is one unit measure of each type of individual. They are all risk neutral and are concerned only with consumption when they are old.[7] We use the feminine pronoun for entrepreneurs and masculine pronoun for both managers and widget producers throughout the paper. Young entrepreneurs are endowed with $I > 0$ units of goods each, while managers and widget producers are endowed nothing. In addition, young entrepreneurs can access storage technology that converts one unit of goods today into $r > 1$ units of goods tomorrow.[8]

There are a continuum of N durable production assets (or projects) in the economy, where $N < 1$. We identify the production assets as "firms" so that each production asset has one firm "name" attached to it. We then treat entrepreneurs who own the production assets as the "owners of firms" who hold the control rights to determine how to use these assets.

We suppose that there is a competitive market, such as a stock market, where entrepreneurs trade the ownership of firms at given market prices. In this market, young entrepreneurs purchase the ownership of a firm from old entrepreneurs at competitive market prices, which we explain in more details below. Then, they run the firm, enter the production process, and earn profits by hiring widget producers and managers when old. Without loss of generality, we suppose that each firm is owned by one entrepreneur.[9] Since widget producers and managers have no endowments when young, they cannot purchase the ownership of a firm.[10]

Firms can produce y_t units of goods using k_t units of widget and managerial capital A_t according to the following production function:

$$y_t = F(A_t, k_t). \tag{1}$$

We assume that F is strictly increasing with $F(A, 0) = F(0, k) = 0$, strictly concave, continuously differentiable, and exhibits constant returns to scale. Here, the managerial capital A_t of a firm includes intangible assets such as accumulated management know-how, knowledge, and skills that successive generations of managers within firms inherit over time.

For production, each firm needs to hire one manager and one widget producer. We

[7]This assumption simplifies the following argument. We can allow individuals to have linear preferences for consumption in both the youth and adult periods.

[8]As an alternative setting, we can consider the following: there is a competitive one-period credit market in which one unit of goods in period t is exchanged for r_t units of goods in period $t+1$. Entrepreneurs born in period t have a linear preference for consumption C_t^y when young and C_{t+1}^o when old. Their utility function is $U_t = C_t^y + \beta C_{t+1}^o$, where $\beta \in (0, 1)$ represents the discount factor. Then, by the utility maximization and equilibrium condition under this linear preference, the equilibrium interest rate r_t is determined by $r_t = r \equiv 1/\beta$.

[9]Since the no-arbitrage conditions hold, as we will see below, the net return on purchasing a firm is zero in equilibrium, so it is optimal for entrepreneurs to own any number of firms.

[10]In addition, widget producers and managers have no incentives to own firms when they are old because they leave the economy, and thus have no opportunity to profit in the future by owning firms.

simplify the firm's hiring process by assuming the existence of a matching market in which firms are matched with young managers and old widget producers in the one-to-one matching manner, without any matching friction.[11] In the matching market, each firm thus surely meets one young manager and one old widget producer. After matching to a firm, an old widget producer produces a widget used to produce the final output. A widget producer can produce widget at a constant marginal cost equal to one in terms of goods.[12] A young manager matched with a firm learns and embodies the managerial capital accumulated in the firm up to the current period. For example, a young manager acquires managerial capital via personal communication and knowledge transfers from the old manager working in the same firm, on-the-job training, and so on. When a manager becomes old in period t, he works in a managerial position and decides how much to contribute to the accumulation of managerial capital by choosing a *management quality* level of $a_t \in \mathbb{A} \equiv \{a^0, ..., a^m\}$, where $a^i > a^{i-1}$ for all i, $m \geq 1$,and $a^0 \geq 0$, at a personal cost (disutility) of $c(a_t)$ measured in goods.[13] We can alternatively interpret a_t as the managerial effort an old manager exerts. Here, we assume that it is more costly for each old manager to choose a higher management quality; that is, $c(a^{i+1}) > c(a^i)$, and that $c(a^0) = 0$. We also assume that the manager's reservation payoff is zero.

An old entrepreneur and a young manager exercise *separation options*. Specifically, at the end of period t, the old entrepreneur who owns a firm can always liquidate the firm's production asset and obtain a liquidation value of $L > 0$. On the other hand, the young manager can leave the firm, and thereby obtain a payoff of zero when he becomes old in the next period, $t + 1$, because he has no opportunities to match to a firm when old.

2.2 Managerial Capital Accumulation

A higher management quality a_t has a larger contribution to managerial capital accumulation. When an old manager chooses a management quality of a_t in period t given the managerial capital of A_{t-1} accumulated up to the previous period $t - 1$, the managerial capital A_t in period t is determined by the following law of motion:

$$A_t = h(a_t, A_{t-1}), \tag{2}$$

where the initial stock of managerial capital A_0 is exogenously given.

[11] We make the assumption of no matching friction only to avoid unnecessary complicated analysis. We can adapt our model to the case in which matching friction exists.

[12] Young widget producers do nothing. Thus, we simply use "widget producers" to refer to old widget producers in the following.

[13] We can extend the model to allow continuous management quality as long as there is some minimum level required to contribute to managerial capital accumulation. Specifically, we can consider the case that $\mathbb{A} = \{0\} \cup [\underline{a}, \overline{a}]$, where $\underline{a} > 0$ is the minimum level of management quality necessary to accumulate managerial capital.

When an old entrepreneur decides to shut down production in period t, she does not hire a matched manager and widget producer. Then, managerial capital does not accumulate in period t at all, in which case we assume that the managerial capital A_t in period t passes to the the next period, $t+1$, according to $A_t = h(0, A_{t-1})$.[14]

We assume that h is continuously differentiable with respect to A_{t-1} and that the marginal effect of the accumulated managerial capital on the current managerial capital is parametrized by an exogenous variable $\delta \in [0,1]$ as follows[15]

$$\eta(a, A; \delta) \equiv \frac{\partial h(a, A)}{\partial A}.$$

We then make the following assumption about the function h.

Assumption 1. *(i) h is strictly increasing in both arguments; concave with respect to the second argument, where $h(0,0) = 0$, $\lim_{A\to\infty} \eta(a, A; \delta) < 1$, and $h(a^0, 0) > 0$; (ii) h has the property of increasing differences; that is, $h(a'', A) - h(a', A)$ is non-decreasing in A for $a'' \geq a'$; and (iii) for any $a \in \mathbb{A}$ and $\varepsilon > 0$, there exists some $\tilde{\delta} \in (0, 1)$ such that for all $\delta \in (\tilde{\delta}, 1)$, we have $|\eta(a, A; \delta) - 1| < \varepsilon$ at any $A > 0$ satisfying $A = h(a, A)$.*[16]

Assumption 1(i) is standard and requires the usual boundary conditions for h to ensure the existence of a unique positive steady state of managerial capital $A > 0$ satisfying $A = h(a, A)$ for any given $a \in \mathbb{A}$.[17] Assumption 1(ii) states that the marginal effect of increasing the management quality on the current managerial capital increases when the firm accumulated more managerial capital in the past. Thus, the steady state capital $A = h(a, A)$ does not decrease as the firm implements higher management quality. Assumption 1(iii) provides a parametric restriction on how the managerial capital accumulated in the past affects the current managerial capital. This marginal effect becomes close to 1 as $\delta \to 1$, which implies that the steady state capital $A = h(a, A)$ diverges to infinity as $\delta \to 1$. We show this result below.

Lemma 1. *The steady state level of managerial capital $A > 0$, which satisfies $A = h(a, A)$ for any given $a \in \mathbb{A}$, diverges to infinity as $\delta \to 1$.*

[14] However, without loss of generality, we can suppose that this case does not occur in equilibrium because entrepreneurs can earn at least some non-negative profits by implementing the lowest management quality, $a^0 \geq 0$.

[15] We restrict the range of δ to $[0,1]$ simply for normalization.

[16] Here, $\tilde{\delta}$ may depend on a and ε.

[17] Since $h(a, 0) > 0$, $h(a, A)$ has a fixed point of A when $h(a, A) < 1$ for a large A. This last point is shown by concavity of h: $h(a, A) - h(a, B) \leq \eta(a, B; \delta)(A - B)$ holds for some fixed B. Then, we have $h(a, A)/A \leq \eta(a, B; \delta)(1 - B/A) + h(a, B)/A$, which goes to $\eta(a, B; \delta) < 1$ as $A \to \infty$ given B. The uniqueness follows from the concavity of h.

The function h that satisfies Assumption 1 includes several useful specifications: (i) $h(a_t, A_{t-1}) = a_t + \delta A_{t-1}$, where $\delta \in (0,1)$ measures how the past managerial capital A_{t-1} depreciates over time. In this case, the steady state of managerial capital is $A = a/(1-\delta)$, which goes to infinity as $\delta \to 1$. Here, note that $\eta(A, a; \delta) = \delta$ is always constant, such that $\eta(A, a; \delta) \to 1$ as $\delta \to 1$. (ii) $h(a, A) = a^{1-\delta}(A+d)^{\delta}$, where $\delta \in (0,1)$ and $d > 0$. In this case, the steady state of managerial capital satisfies $A = a^{1-\delta}(A+d)^{\delta}$, which goes to infinity as $\delta \to 1$.

2.3 Firm Ownership Market

In period t, young entrepreneurs purchase the ownership of a firm from old entrepreneurs at the competitive market price of V_t (stock price in period t). They expect to earn the following rate of return on owning a firm:

$$\frac{\pi_t + \max\{V_t, L\}}{V_{t-1}} \tag{3}$$

when they purchase it at the market price of V_{t-1} in period $t-1$. Here, π_t denotes the flow profit (dividend) of the firm. Young entrepreneurs expect to earn V_t by selling the firm ownership to the next generation when they become old in the next period t. The old entrepreneur who owns a firm can always liquidate the firm to earn $L > 0$ to secure at least $\max\{V_t, L\}$. By the no arbitrage condition, the above rate of return on holding a firm ownership must be equal to the rate of return on storage technology r in equilibrium:

$$rV_{t-1} = \pi_t + \max\{V_t, L\}. \tag{4}$$

3 Limited Commitment and Self-Enforcing Contracts

3.1 Verifiability, Observability, and Separation Option

We assume that both an old entrepreneur and a young manager matched in period t can observe the management quality, a_t, chosen by the old manager and the widget production level k_t chosen by a hired widget producer within the same firm at the end of period t. However, we assume that outside parties, particularly courts, cannot verify these choices, a_t and k_t, and hence they are not formally contractible. We also assume that outside parties cannot verify the production output y_t, the accumulated managerial capital A_t, and endowment of entrepreneurs I. Thus, old entrepreneurs may renege on agreed upon payments to hired old managers and widget producers after they choose the management quality and widget production levels. We next see how old entrepreneurs can commit to contracts in the self-enforcing manner.

As we noted, the matched old entrepreneur and young manager have separation options at the end of each period. Specifically, we suppose that they simultaneously decide whether or not to exercise their separation options after they observe how much the old entrepreneur pays to the widget producer and the old manger. We also assume that, although the old manager's chosen management quality and the widget producer's chosen widget production are not public information; all young entrepreneurs entering the economy in period t can observe whether or not the young manager hired by the firm left the firm at the end of period t before they decide to purchase the ownership of the firm.

3.2 Dynamic Enforcement Constraint

Although it is impossible to write formal contracts contingent on management quality a_t and widget production k_t because they are not verifiable, each old entrepreneur can offer informal contracts, $\{R_t, k_t\}$ to a hired widget producer and $\{b_t, a_t\}$ to a hired old manager, in the beginning of period t. Here, the informal contract $\{R_t, k_t\}$ specifies the production level of widget k_t and the corresponding payment R_t to the widget producer, while $\{b_t, a_t\}$ specifies the management quality a_t for the old manager and the corresponding wage b_t. For the widget producer and old manager to accept these informal contracts, they must satisfy the following individual rationality (IR) constraints:

$$R_t - k_t \geq 0 \tag{IR-W}$$

$$b_t - c(a_t) \geq 0, \tag{IR-M}$$

where the widget producer and the old manager obtain the reservation payoffs normalized to zero when they reject the old entrepreneur's offered informal contracts. Here, $R_t \geq k_t \geq 0$ implies that $R_t \geq 0$. Also $b_t \geq c(a_t) \geq 0$ implies that $b_t \geq 0$.

The events in each period are as follows.

Period t:

1. Firms operated by old entrepreneurs are matched with young managers and widget producers.

2. Each old entrepreneur offers a wage contract $\{b_t, a_t\}$ to the old manager retained from the previous period $t-1$ and makes a contract with a matched widget producer to produce k_t units of widgets in exchange for a payment of R_t. Both the manager and the widget producer decide whether or not to accept these contracts. When the widget producer rejects the contract, there is no productions of the final output because it is essential to production ($F(A_t, 0) = 0$), and then all parties obtain their reservation payoffs. If the old manager rejects the contract while the widget producer accepts it, production takes place given the managerial capital $A_t = h(0, A_{t-1})$.[18]

[18] However this case never happens in equilibrium.

3. The old manager chooses a management quality level of $a_t \in \mathbb{A}$. The widget producer produces k_t units of widgets. Then, the firm produces a final output of y_t by combining k_t units of widgets and managerial capital $A_t = h(a_t, A_{t-1})$.[19]

4. The old entrepreneur decides whether or not to pay the agreed upon payments b_t and R_t to the old manager and the widget producer, respectively. The young manager hired by the same firm observes these payments.

5. The young manager and the old entrepreneur simultaneously decide whether or not to exercise their separation options.

6. When the young manager decides to stay in the firm, he embodies the managerial capital A_t accumulated up to the current period t through knowledge transfers from the old manager of the firm.

7. Young entrepreneurs who purchased the ownership of a firm from old entrepreneurs will run the firm in the next period, $t+1$, according to the same timing as in period t.

Now we derive the condition for which an old entrepreneur optimally honors the informal contract $\{R_t, k_t, b_t, a_t\}$ with an old manager and a widget producer in period t. Suppose that the old manager and the widget producer chose a_t and k_t, respectively, as specified in the informal contract in period t. Then, we check the incentive for the old entrepreneur to honor the payments of R_t and b_t according to the informal contract $\{R_t, k_t, b_t, a_t\}$.

By exercising the separation option, each old entrepreneur can always renege on payments b_t and R_t, and then liquidate the firm's production asset to earn at least the payoff of $y_t + L$.[20] On the other hand, by honoring the informal contract agreement $\{R_t, b_t, a_t, k_t\}$ and selling the ownership of the firm to a young entrepreneur at its market price of V_t, the old entrepreneur would obtain an equilibrium payoff of $y_t - (b_t + R_t) + \max\{V_t, L\}$. Thus, for an old entrepreneur not to renege on b_t or R_t, $\max\{V_t, L\} - L \geq b_t + R_t$ must hold.

When $V_t < L$ occurs in some period t, this condition is satisfied only when $b_t = R_t = 0$ because $b_t \geq 0$ and $R_t \geq 0$. However, $R_t = 0$ implies that $k_t = 0$, and hence $y_t = F(A_t, 0) = 0$, such that the firm's profit in period t must be zero, $\pi_t = 0$. From the no-arbitrage condition (4), it then follows that $V_{t-1} = \max\{V_t, L\}/r = L/r$, which is less than L; that is,

[19] Here, the incentive problem occurs in terms of the old manager's choice of management quality level a_t, but there are no moral hazard problems regarding the provision of managerial capital A_t once the manager acquires management quality a_t. The old manager embodies the managerial capital A_t accumulated in period t, which contributes to the production of the final output of y_t when the old manager is in the management position.

[20] More precisely, an old entrepreneur's total payoff in period t is $\pi_t + V_t + r(I - V_{t-1})$, where the last term is the income from investing $I - V_{t-1}$ in the storage technology after purchasing the firm ownership in the previous period $t-1$. However, this is a sunk cost when entrepreneurs are old, so we ignore it when considering their incentives.

$V_{t-1} < L$ holds. Repeating this, $V_0 < L$ holds so that old entrepreneurs in the initial period $t = 0$ liquidate their firms and there is no production in the subsequent periods. Although this becomes a trivial equilibrium path, we will not focus on this equilibrium, but rather on the more interesting equilibrium in which $V_t \geq L$ for all $t \geq 0$, and thus no firms are liquidated in the initial period. In the following, when we refer to an equilibrium, we mean a non-trivial equilibrium.

In a non-trivial equilibrium, an old entrepreneur self-enforces the informal contract only if the *dynamic enforcement (DE)* constraint is satisfied:

$$V_t - L \geq R_t + b_t. \tag{DE}$$

DE is a necessary condition to make the informal contract $\{R_t, k_t, b_t, a_t\}$ self-enforcing. Conversely, we can show that DE constraint becomes sufficient for an old entrepreneur to honor the informal contract $\{R_t, k_t, b_t, a_t\}$.

Lemma 2. *Each old entrepreneur honors the informal contract* $\{R_t, k_t, b_t, a_t\}$ *in period t if and only if DE is satisfied.*

Intuitively, when DE is satisfied, an old entrepreneur who owns a firm cannot gain from any deviation from the agreed upon payments of R_t and b_t after the widget producer and the old manager choose k_t and a_t, respectively, according to the informal contract, provided the deviation leads to a low market value of the firm in future periods. Our construction of the continuation equilibrium that triggers this punishment is similar to that of Kreps (1990), where once a firm loses its reputation, it will not be sold at a high market price in any future period, which then identifies the deviation by a firm's current owners. In our context, when an old entrepreneur who owns firm i reneges on R_t or b_t in period t, this deviation results in a continuation equilibrium in which the young manager hired by firm i and the old entrepreneur simultaneously exercise their separation options in the same period given the belief that future old entrepreneurs who will own firm i and future young managers who will be hired by firm i will exercise their separation option after observing that young managers quit firm i at the end of each period. Anticipating the continuation equilibrium in which the firm will be liquidated in the future period, the old entrepreneur who deviates from the informal contract in the current period t will liquidate firm i to earn the liquidation value of L. Thus, the old entrepreneur obtains at most $y_t + L$ by any deviation from the informal contract, which is however lower than the equilibrium payoff of $y_t - (b_t + R_t) + V_t$ when DE is satisfied.

3.3 Optimal Self-Enforcing Contracts

We focus on the non-trivial equilibrium in which old entrepreneurs choose optimal contracts to maximize their profits from among all self-enforcing contracts. Specifically, each old entrepreneur in period t offers a self-enforcing contract $\{R_t, k_t, b_t, a_t\}$ to maximize her profit, as follows.

Problem P.

$$\max_{R_t, k_t, b_t, a_t} F(h(a_t, A_{t-1}), k_t) - R_t - b_t$$

subject to IR-M, IR-W, and DE, given V_t and A_{t-1}.

Here, old entrepreneurs take A_{t-1} and V_t as exogenously given because the former is predetermined in the previous period and the later is determined in the competitive market in which everyone takes V_t as given.

Since both IR-M and IR-W are binding at the optimum, we can write Problem P as

Problem P

$$\max_{k_t, a_t} F(h(a_t, A_{t-1}), k_t) - k_t - c(a_t)$$

subject to

$$V_t - L \geq k_t + c(a_t) \tag{DE}$$

given A_{t-1} and V_t.

The important feature of Problem P is that DE constrains both the choice of management quality a_t and widget production level k_t. In particular, when DE is binding, the old entrepreneur must reduce k_t (or a_t) in order to increase a_t (or k_t). In this sense, management quality a_t and widget production k_t become substitutes through the DE constraint. This will play a key role in causing the management cycles, as we will see below.

We define the maximum profit of an old entrepreneur who solves Problem P, given a management quality a_t, a predetermined managerial capital A_{t-1}, and a firm market value of V_t, as follows

$$\tilde{\Phi}(a_t; A_{t-1}, V_t) \equiv \max_{k_t : V_t - L - c(a_t) \geq 0} F(h(a_t, A_{t-1}), k_t) - k_t - c(a_t), \tag{5}$$

provided $V_t \geq L + c(a_t)$ (otherwise, no optimal solutions exist). For convenience, we define $\tilde{\Phi}(a_t; A_{t-1}, V_t) = 0$ when $V_t - L < c(a_t)$.

When DE is not binding, $\tilde{\Phi}(a_t; A_{t-1}, V_t)$ is equal to $\max_{k_t} F(h(a_t, A_{t-1}), k_t) - k_t - c(a_t)$, which does not vary with the firm value V_t. However, when DE is binding, we obtain

$$\tilde{\Phi}(a_t; A_{t-1}, V_t) = F(h(a_t, A_{t-1}), V_t - L - c(a_t)) - (V_t - L), \tag{6}$$

which is continuous and increasing in V_t. Note that $\Phi(a_t; V_t, A_{t-1}) = 0$ for all $V_t < c(a_t) + L$.

Given the profit function (5) above, each old entrepreneur chooses the management quality a_t to maximize her profit $\tilde{\Phi}(a_t; A_{t-1}, V_t)$ from all possible management quality levels $\mathbb{A} \equiv \{a^0, ..., a^m\}$, given V_t and A_{t-1}. We then define the overall maximum profit each old entrepreneur attains in period t as

$$\Phi(V_t, A_{t-1}) \equiv \max_{a \in \mathbb{A}} \tilde{\Phi}(a; A_{t-1}, V_t). \tag{7}$$

Since $\tilde{\Phi}$ is continuous in V_t and A_{t-1}, and the set of all possible management qualities $\mathbb{A}$ is finite, Φ is continuous in V_t and A_{t-1}. We denote by $\hat{a}(V_t, A_{t-1})$ the optimal management quality that attains the above maximum profit (7).

4 Equilibrium Dynamics with Management Cycles

4.1 Equilibrium Conditions

Since we showed that (7) determines the flow profit a firm can attain in period t, we obtain $\pi_t = \Phi(V_t, A_{t-1})$. By substituting this into (4), we obtain

$$rV_{t-1} = \Phi(V_t, A_{t-1}) + V_t. \tag{8}$$

Additionally, since the optimal management quality a_t in period t is given by $a_t = \hat{a}(V_t, A_{t-1})$, the evolution of managerial capital A_t follows according to

$$A_t = h(\hat{a}(V_t, A_{t-1}), A_{t-1}), \tag{9}$$

given the exogenous initial condition A_0.

Equations (8) and (9) fully govern the dynamics of the equilibrium values of A_t and V_t, which we analyze in the following.

4.2 Benchmark: Non-Binding DE

Before showing the main result, we first consider the benchmark case in which V_t is so large that we can ignore the DE constraint in Problem P. In this benchmark, the old entrepreneur chooses the widget production level k_t^* to maximize her profit $F(h(a_t, A_{t-1}), k) - k$, which yields $k_t^* = \lambda A_t$ for $A_t = h(a_t, A_{t-1})$ due to the constant returns to scale of F, where λ maximizes $F(1, k) - k$ over $k \geq 0$.

The maximum profit in the benchmark case is then

$$\pi^*(A_{t-1}) \equiv \max_{a_t \in \mathbb{A}} \ h(a_t, A_{t-1})\psi - c(a_t),$$

where $\psi \equiv F(1, \lambda) - \lambda > 0$. We denote by $M(A_{t-1})$ the set of optimal management quality levels that achieve the maximum level above. We also denote by $a^*(A_{t-1}) \in M(A_{t-1})$ an optimal choice that satisfies

$$h(a^*(A), A)\psi - c(a^*) \geq h(a, A)\psi - c(a) \quad \text{for all } a \neq a^*(A). \tag{10}$$

Given the result above, the market value of the firm V_t evolves according to $rV_{t-1} = \pi^*(A_{t-1}) + V_t$ when we can ignore DE.

Since h has the property of increasing differences under Assumption 1, we can verify that $\min M(A'') \geq \max M(A')$ for any $A'' > A'$. Then, for any $a^*(A) \in M(A)$, we can show that $h(a^*(A), A)$ is non-decreasing in A under Assumption 1.[21]

Then, the managerial capital in the benchmark case evolves according to

$$A_t = h(a^*(A_{t-1}), A_{t-1}),$$

from which we can show that any equilibrium path of $\{A_t\}_{t=0}^{\infty}$ is monotonic. To see this, suppose that $A_{t-1} \geq A_{t-2}$. Then, we obtain $A_t = h(a^*(A_{t-1}), A_{t-1}) \geq h(a^*(A_{t-2}), A_{t-2}) = A_{t-1}$, implying that $A_t \geq A_{t-1}$. Repeating this, A_t is monotone non-decreasing when $A_1 > A_0$. On the other hand, if $A_1 < A_0$, then any equilibrium sequence of A_t is monotone non-increasing. This shows that a cyclical equilibrium path in the benchmark case never exists. The crucial part of the benchmark is that the optimal choice of management quality $a^*(A_{t-1})$ does not depend on the market value of firm V_t. Thus, the market value of firm V_t never affects the evolution of managerial capital, which is in sharp contrast to the case below in which DE is binding.

Even when we introduce DE, it can be slack when the firm's market value V_t is large enough. This occurs when k_t^* and $a^*(A_{t-1})$ satisfy DE:

$$\begin{aligned} V_t &\geq V^*(A_{t-1}) \\ &\equiv k_t^* + c(a^*(A_{t-1})) + L \\ &= \lambda h(a^*(A_{t-1}), A_{t-1}) + c(a^*(A_{t-1})) + L \end{aligned}$$

From this observation, we can depict the equilibrium condition (8) in the plane of V_t and V_{t-1}, given A_{t-1}. First, note that since the firm's profit $\Phi(V_t, A_{t-1})$ becomes $\pi^*(A_{t-1})$ for all $V_t > V^*(A_{t-1})$, in which case, DE is not binding, we have $(1/r)\{\Phi(V_t, A_{t-1}) + V_t\} = (1/r)\{\pi^*(A_{t-1}) + V_t\} < V_t$ for all $V_t > V^*(A_{t-1})$. Then, the curve $(1/r)\{\Phi(V_t, A_{t-1}) + V_t\}$ can

[21] For $A'' > A'$, we obtain $h(a^*(A''), A'') \geq h(\min M(A''), A'') \geq h(\max M(A'), A') \geq h(a^*(A'), A')$.

be below the 45 degree line for all large $V_t > V^*(A_{t-1})$. On the other hand, when V_t is small enough, we already know that $\Phi(V_t, A_{t-1}) = 0$, in which case $V_{t-1} = (1/r)V_t$ holds. These facts, together with continuity of $\Phi(\cdot; A)$ with respect to V, show the existence of the largest and smallest values of V, denoted by $\overline{V}(A)$ and $\underline{V}(A)$, such that $rV = \Phi(V, A) + V$. Then, $\underline{V}(A) > 0$ holds when A is not so small (see Figure 2(a) for the case of a large $A_{t-1} = A$). When A is small enough, $(1/r)\{\Phi(V, A) + V\}$ is always below the 45 degree line (see Figure 2(b) for the case of a small $A_{t-1} = A$).

If the maximum profit in the benchmark in the absence of DE becomes zero in period t, equilibrium profits in the presence of DE must be always zero as well. To avoid this uninteresting case, we maintain the following assumption throughout the paper to ensure that the firm's profit can be positive, at least when DE is slack.

Assumption 3. $\max_{a \in \mathbb{A}} h(a, 0)\psi - c(a) > 0$.

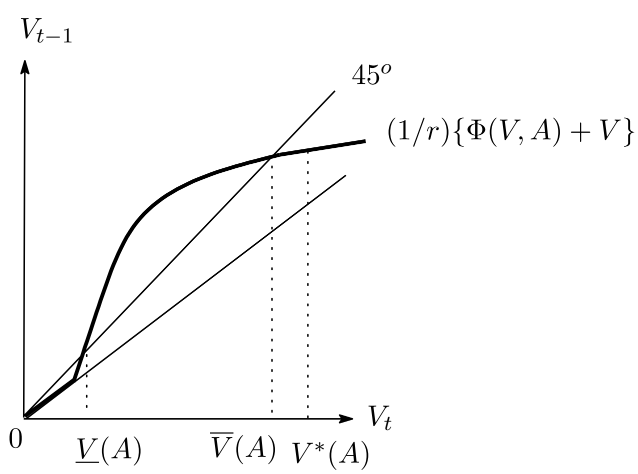

Figure 2(a): Case of large $A_{t-1} = A$

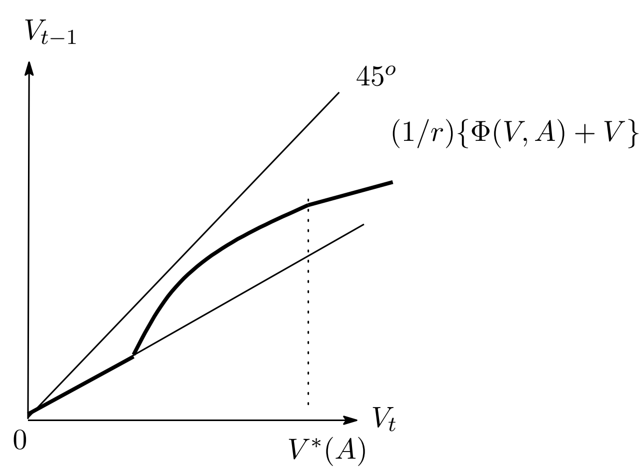

Figure 2(b): Case of small $A_{t-1} = A$

4.3 Cyclical Equilibrium Paths

Next we show our main result that when DE is a tight constraint, the equilibrium paths show a cyclical feature: firms switch between different management quality levels over time.

An equilibrium path of the economy is given by a sequence, $\{V_t, A_t\}_{t=0}^{\infty}$, of the firm's market value V_t and managerial capital A_t, which satisfy both (8) and (9), as well as $V_t \in [L, I]$, where the initial value of managerial capital A_0 is exogenously given. Since young entrepreneurs cannot purchase the ownership of a firm when its price exceeds their endowment I, we need $V_t \leq I$. Moreover, in the non-trivial equilibrium that we focus on, the firm's value must satisfy $V_t \geq L$. Note also that the initial market value of the firm, V_0, is a jump variable that is endogenously determined in equilibrium.

We say that an equilibrium path $\{V_t, A_t\}_{t=0}^{\infty}$ is *convergent in management quality* if there exists some management quality $a^i \in \mathbb{A}$ and some period T such that for all $t \geq T$, $a_t = a^i$ holds. That is, firms sustain the same management quality a^i forever from some period T onward. In particular, when $a_t = a^i$ and $a^i \neq a^0$ for all $t \geq T$, firms sustain some "good" management quality level of $a^i \neq a^0$ in the long run. On the other hand, when $a_t = a^0$ for all $t \geq T$, firms are stuck at the lowest management quality a^0 in the long run. When an equilibrium is not convergent in management quality, since the set of possible management quality levels $\mathbb{A}$ is finite, for any period t, there exist some periods $t'' > t' > t$ such that $a_{t'} \neq a_t$ and $a_{t''} = a_t$ hold.[22] We call this equilibrium *cyclical equilibrium* in the sense that it involves infinite switches between management quality levels over time.

We define the steady state value of managerial capital in an equilibrium which is convergent in management quality a^i, as follows

$$A^i = h(a^i, A^i). \tag{11}$$

This steady state always exists and is unique under Assumption 1. In particular, the lowest steady state $A^0 = h(a^0, A^0)$ converges to zero as $a^0 \to 0$ under Assumption 1.

Recall that the parameter value δ represents the effect of the managerial capital accumulated in the past period on the current managerial capital. We then show our main result as follows.

Proposition 1. *There exist some $\overline{\delta} \in [0,1)$ and $\overline{a}^0 > 0$ such that for all $\delta \in [\overline{\delta}, 1)$ and all $a^0 \in [0, \overline{a}^0]$, every equilibrium path becomes cyclical.*

The intuition behind Proposition 1 is as follows. To clarify, we consider an example in which A_t evolves according to $A_t = a_t + \delta A_{t-1}$ for $\delta \in (0,1)$ where we interpret $1-\delta$ as the rate of depreciation of managerial capital. In this simple case, the steady state level of managerial capital when the firm implements the management quality level a^i is $A^i \equiv a^i/(1-\delta)$, which goes to infinity as $\delta \to 1$.

If there exists some equilibrium path that is convergent in a "good" management quality level for $a^i \neq a^0$, then firms induce hired managers to choose a^i from some period onward. When the past managerial capital has a large impact on the accumulation of the current managerial capital (this is true when it does not depreciate faster; that is, δ is close to 1), the managerial capital eventually becomes sufficiently large in the long run; that is,

[22] In any equilibrium that is not convergent in management quality, for any period t, there must exist some periods $t_2 > t_1 \geq t$ such that $a_{t_2} \neq a_{t_1}$. Moreover, there must exist some period $t_3 > t_2$ such that $a_{t_3} \neq a_{t_2}$. Repeating this, some period $\tilde{t}$ must exist such that $a_{\tilde{t}} = a_{t_1}$.

$A_t \to A^i \equiv a^i/(1-\delta) \to \infty$ as $\delta \to 1$. In Figure 3 (a), we depict how the firm's market value V_t evolves over time given managerial capital fixed at the long-run level of A^i.[23] In this case, the firm's profit $\Phi(V_t, A^i)$ becomes large enough so that the market value of the firm, V_t, must converge to some finite value which is less than I, denoted by $\underline{V}^i$ in Figure 3 (a), because any other path of V_t eventually exceeds the entrepreneur's endowment, I. Here, $\underline{V}^i > c(a^i) + L$ must hold in this equilibrium; otherwise, no feasible contracts exist in Problem P. Additionally, since V_t is close to $\underline{V}^i$ as $t \to \infty$, the no-arbitrage condition (8) implies that the equilibrium profit approaches $\Phi(\underline{V}^i, A^i) = (r-1)\underline{V}^i$.

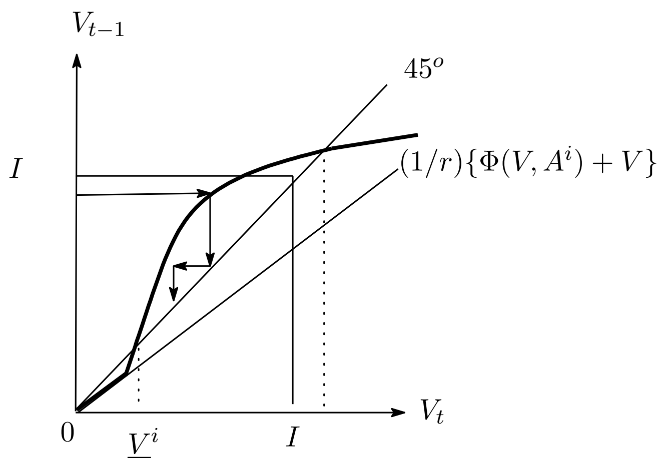

Figure 3(a): Case of $A_t \to A^i$

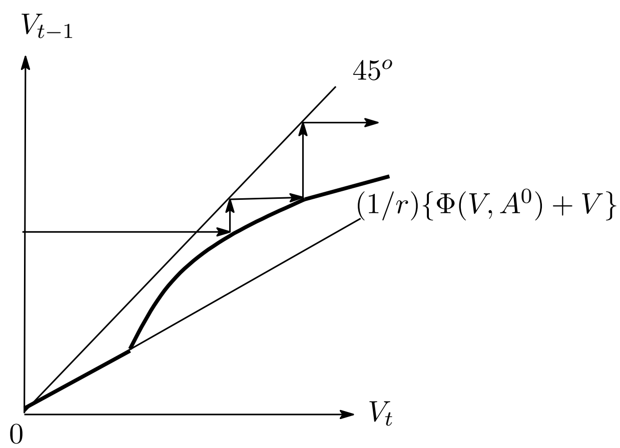

Figure 3(b): Case of $A_t \to A^0$

Provided that enough managerial capital $A_{t-1} \simeq A^i$ accumulates and that the firm's market value of V_t is close to $\underline{V}_i > c(a^i) + L$, suppose that some old entrepreneur deviates to implement the lowest management quality level, a^0, from an old manager and chooses a widget production level of $k_t = V_t - c(a^0) - L$. Since $V_t \simeq \underline{V}^i > c(a^i) + L > c(a^0) + L$, the deviating entrepreneur can obtain at least a profit of

$$F\left(h(a^0, A^i), \underline{V}^i - c(a^0) - L\right) - (\underline{V}_i - L)$$

when $V_t \to \underline{V}^i$ and $A_t \to A^i$ as $t \to \infty$. This profit can be large enough when $\delta \to 1$, so $A^i \to \infty$. Then, this deviation profit is strictly larger than the equilibrium profit $\Phi(\underline{V}^i, A^i) = (r-1)\underline{V}^i$. Intuitively, the current entrepreneurs have no incentives to increase management quality further when the firm accumulates enough managerial capital because its marginal product $\partial F/\partial A$ is decreasing in A_t. Rather, since the implementation of management quality and widget production become substitute via the DE constraint, entrepreneurs reduce the

[23] This does not correctly depict the entire dynamics of both state variables V_t and A_t because A_t approaches A^i, but it does not exactly equal A^i. We use Figure 3 only to illustrate the rough intuition of Proposition 1.

management quality, but increase widget production when the firm accumulates enough managerial capital. Thus, the firm does not sustain any good management quality $a^i \neq a^0$ in the long run.

On the other hand, if an equilibrium path that is convergent in the lowest management quality a^0 exists, then managerial capital eventually converges to $A^0 = h(a^0, A^0)$. However, when the lowest management quality a^0 has only a small contribution to the accumulation of managerial capital ($a^0 \simeq 0$), the firm's equilibrium profit $\Phi(V_t, A_{t-1})$ becomes sufficiently close to a low value of $\Phi(V_t, A^0)$ in the long run. Unless such a low profit is offset by the increase in its future market value, no young entrepreneurs purchase the ownership of the firm in the current period. Thus, to keep the no-arbitrage condition (8), the firm's market value must increase (see Figure 3 (b)). This makes DE easier to satisfy, so if some old entrepreneur deviates from the equilibrium choice a^0 and instead implements a management quality of $a_t \neq a^0$, she can raise her profits, thus contradicting the optimality of implementing the lowest management quality of a^0.

These arguments establish that every equilibrium must be cyclical, provided $\delta \to 1$ (a low depreciation rate of managerial capital) and $a^0 \to 0$ (the lowest management quality a^0 has a small contribution).

4.4 Existence of Cyclical Equilibrium Paths

We showed that every equilibrium must be cyclical when the managerial capital accumulated up to the previous period has a large impact on its current level and implementing the lowest management quality level contributes little to its accumulation.

Next, under these conditions, we show that a cyclical equilibrium path actually exists.

Recall that A^i denotes a steady state capital $A^i = h(a^i, A^i)$ when management quality a^i is sustained forever. When the initial value of managerial capital A_0 satisfies $A_0 \in (A^0, A^m)$, A_t remains within the region (A^0, A^m) for all $t \geq 0$.[24] In the following, we assume that $A_0 \in (A^0, A^m)$, implying that $A_t \in (A^0, A^m)$ for all $t \geq 0$. Then, we obtain

$$rV_{t-1} \geq \Phi(V_t, A^0) + V_t \tag{12}$$

and

$$rV_{t-1} \leq \Phi(V_t, A^m) + V_t \tag{13}$$

because the firm's profit Φ is non-decreasing in A_{t-1}. We then define the firm's smallest and largest market values, $\underline{V}^i$ and $\overline{V}^i$, $i = 0, m$, for the smallest and largest management quality levels a^0 and a^m, respectively, as the value of V that satisfies

$$rV = \Phi(V, A^i) + V, \quad i = 0, m. \tag{14}$$

[24] If $A_{t-1} \in (A^0, A^m)$, then $A_t \leq h(a^m, A_{t-1}) \leq h(a^m, A^m) = A^m$. Additionally, $A_t \geq h(a^0, A_{t-1}) \geq h(a^0, A^0) = A^0$.

We provide the conditions under which such values exist and satisfy $\underline{V}^i > 0$ for $i = 0, m$. To this end, we make the following assumption:

Assumption 4. *There exists some $a^i \neq a^0$ such that*

$$\max_{V \geq c(a^i)+L} F(h(a^i, A^0), V - L - c(a^i)) - rV + L > 0.$$

Assumption 4 ensures the existence of some management quality level of $a^i \neq a^0$ such that in a stationary environment with a constant firm value V, the firm's profits plus capital gains from selling the firm ownership can outweigh the opportunity cost of purchasing the firm ownership, rV, even when the managerial capital accumulated up to the previous period is small and is close to the lowest steady state A^0. To see this more precisely, note that each old entrepreneur can gain at least the profit of $F(h(a^i, A), V - L - c(a^i)) - (V - L)$ by choosing a widget production level of $k = V - L - c(a^i)$, making DE binding when $V > L + c(a^i)$. This profit plus the capital gains V equals $F(h(a^i, A), V - L - c(a^i)) + L$, which is larger than the opportunity cost of purchasing the firm ownership rV under Assumption 4.

Assumption 4 is satisfied when there exists some management quality $a^i \neq a^0$ which significantly increases the firm's output $F(h(a^i, A^0), k)$ even at the lowest steady state capital A^0.

We show the following lemma.

Lemma 3. *Under Assumption 4*

$$G(V; a) \equiv \max_{k: V - L - c(a_t) \geq k} F(h(a, A^0), k) - k - c(a) - (r - 1)V > 0$$

when $V \geq L + c(a_t)$.

Thanks to Lemma 3, we can ensure that there exist the largest and smallest values of V, denoted by $\overline{V}^0$ and $\underline{V}^0$, respectively such that $rV = \Phi(V, A^0) + V$ and $\underline{V}^0 > 0$. In addition, since $\Phi(V, A^m) \geq \Phi(V, A^0)$ holds for all $V > L + c(a^m)(> L + c(a^0))$, the largest and smallest values of V, denoted by $\overline{V}^m$ and $\underline{V}^m$, respectively, exist such that $rV = \Phi(V, A^m) + V$ and $\underline{V}^m > 0$ as well. Note that $\overline{V}^0 > \underline{V}^m$ is satisfied because $\Phi(V, A^m) > \Phi(V, A^0)$ for all $V \geq \underline{V}^0$ (see Figure 4).

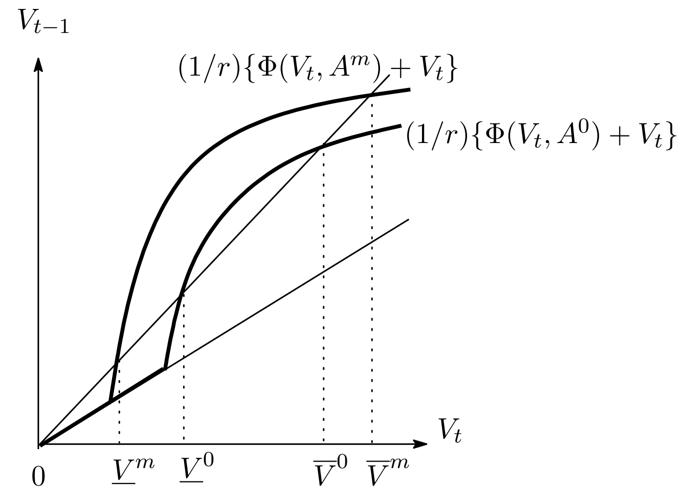

Figure 4: Equilibrium market values of firms

To guarantee that young entrepreneurs own a large enough endowment to purchase the firm ownerships in equilibrium, we make the following assumption.

Assumption 5. $I > \overline{V}^0$.

We then show the existence of a cyclical equilibrium path.

Proposition 2. *Suppose that Assumptions 1-5 hold and that $A_0 \in (A^0, A^m)$. Then, there exist some $\overline{\delta} \in (0, 1)$ and $\overline{a}^0 > 0$ such that for all $\delta \in (\overline{\delta}, 1)$ and all $a^0 \in [0, \overline{a}^0)$, a cyclical equilibrium path exists.*

We obtain this result as follows. As we noted, as long as the initial value of managerial capital A_0 lies in the interval (A^0, A^m), it stays there forever. Given this, the firm's profit $\Phi(V_t, A_{t-1})$ is bounded above and below by $\Phi(V_t, A^m)$ and $\Phi(V_t, A^0)$, respectively, in any period t. These results imply (13) and (14), from which we can show that $\overline{V}^0$ and $\underline{V}^m$ become the upper and lower bounds for the firm's market value, V_t. In fact, the firm's market value V_t remains in the region $(\underline{V}^m, \overline{V}^0)$ as long as its initial value V_0 lies in that region. Although A_0 is exogenously given, V_0 is an endogenous object so that we can actually take an initial value of $V_0 \in (\underline{V}^m, \overline{V}^0)$, which results in $V_t \in (\underline{V}^m, \overline{V}^0)$ for all $t \geq 0$. We can use this result and Proposition 1 to verify that every path $\{V_t\}_{t=0}^{\infty}$ that starts with $V_0 \in (\underline{V}^m, \overline{V}^0)$ must be cyclical, provided $\delta \to 1$ and $a^0 \to 0$.

5 Management Cycles and Profit Volatility

We next show that a firm's profits π_t never converge, but rather increase and decrease along with management cycles in equilibrium. As mentioned in the Introduction, many corporations suffered from large swings in profits over time due to the loss of managerial fidelity through corporate scandals and the recovery from the scandals. Thus, management successes and failures are associated with the rise and fall of profits over time. Moreover, corporate profits show a larger volatility than real GDP does in the US (Figure 1), and the model that incorporates a firm's intangible assets (Hu and Johri 2018) may explain this high volatility in profits well.

We suppose that δ is close to unity and a^0 is small enough such that every equilibrium path becomes cyclical (Proposition 1). Then, we first show that a firm's profits never converge, but rather fluctuate over time in any equilibrium path.

Proposition 3. *Suppose that δ is close to 1 and a^0 is small enough. Then, there exists some $\varepsilon > 0$ such that for any $T \geq 1$, we have $\pi_{t+1} - \pi_t > \varepsilon$ for some $t \geq T$ and $\pi_{s+1} - \pi_s < -\varepsilon$ for some $s \geq T$, where $t \neq s$.*

Proposition 3 shows that a firm's profits π_t rise and fall in infinitely many periods in any equilibrium path. This result comes from the dynamic mechanism showed in Proposition 1: suppose that profit differences $\Delta\pi_{t+1} \equiv \pi_{t+1} - \pi_t$ converge to zero as $t \to \infty$. Then, since the difference in firm value $\Delta V_t \equiv V_t - V_{t-1}$ satisfies $r\Delta V_{t-1} = \Delta\pi_t + \Delta V_t$ due to the no-arbitrage condition (8), ΔV_t must converge to zero as $t \to \infty$. This implies that managerial capital A_t also converges because $\pi_t = \Phi(V_t, A_{t-1})$ holds. However, this contradicts Proposition 1, which states that the management quality level a_t. Hence the profit differences $\Delta\pi_t$, never converge. By combining this with the fact that $\{\Delta\pi_t\}$ is a bounded sequence, $\{\Delta\pi_t\}$ is not monotonic.

We next use Proposition 3 to show that firms' profits change more drastically over time than the real GDP of the economy does in any equilibrium path. This is consistent with the aforementioned empirical finding. To this end, we define the real GDP of our model economy as follows: young entrepreneurs born in period t earn an endowment of I units of goods and spend V_t to purchase the ownership of a firm from an old entrepreneur. Thus, they obtain a net income of $I - V_t$. On the other hand, old entrepreneurs in period t earn a profit of π_t and the value of the firm V_t by selling the firm's ownership to a young entrepreneur. They also earn an income of $r(I - V_{t-1})$ from their investment $I - V_{t-1}$ in the storage technology when they were young in the previous period $t-1$. Thus, old entrepreneurs obtain a total income of $r(I - V_{t-1}) + \pi_t + V_t$. Old managers and widget producers receive b_t and R_t in period t. Then, since $\pi_t = y_t - b_t - R_t$, the total income of our model economy in period t

is the sum of all of these incomes; that is, $\pi_t + r(I - V_{t-1}) + V_t + I - V_t + b_t + R_t$. This then provides the real GDP, denoted by Y_t:

$$Y_t \equiv r(I - V_{t-1}) + y_t + I \tag{15}$$

We measure the volatility of real GDP as its variance from the initial period to period T as

$$\text{Var}_T(Y) \equiv (1/T)\sum_{t=1}^{T}\left(Y_t - (1/T)\sum_{s=1}^{T} Y_s\right)^2. \tag{16}$$

Similarly, we measure the volatility of aggregate firm profits π_t as their variance from the initial period to period T as

$$\text{Var}_T(\pi) \equiv (1/T)\sum_{t=1}^{T}\left(\pi_t - (1/T)\sum_{s=1}^{T} \pi_s\right)^2. \tag{17}$$

Then we show the following result.

Proposition 4. *Suppose that Assumptions 1-3 are satisfied. Suppose also that $\delta \to 1$ and $a^0 \to 0$, and that $A_0 \in (A^0, A^m)$. Then, in any equilibrium path, there exists some period T^* such that for all $T \geq T^*$, we have $Var_T(\pi) > Var_T(Y)$.*

The intuition is as follows. When DE is binding, we obtain $V_t - L = b_t + R_t$ so that $\pi_t = y_t - (R_t + b_t) = y_t - V_t + L$. Then, no-arbitrage condition (8) implies that $rV_{t-1} = \pi_t + V_t = y_t + L$, which yields $Y_t = (r + I) - L$. Thus, the real GDP Y_t becomes a constant in period t when DE becomes binding. As we showed in Proposition 1, every equilibrium path becomes cyclical when $\delta \to 1$ and $a^0 \to 0$. In such a cyclical equilibrium path, DE must be binding for infinitely many periods. Since the real GDP becomes constant for infinitely many periods, while the aggregate firm profits π_t fluctuate over time due to Proposition 3, the latter can show higher volatility than does the former.

6 An Extension: Unobservable Actions and Relational Incentive Pay

We so far considered the model in which all relevant parties within a firm can observe the old manager's choice of management quality, though an outside party cannot verify this information. However, when we interpret the choice a_t as an old manager's managerial effort, it might be reasonable to assume that this is observable to only the manager making the choice. In this section, we discuss an extension of the model to the case in which the management quality chosen by the old manager, a_t, is his own private information, and the old entrepreneur and widget producer within the same firm can observe an informative, but unverifiable, signal of the manager's choice. In this case, the old manager's wages cannot be contingent on their actions directly, but on the observable signals. However, since such signals are not verifiable, wage contracts with old managers must be self-enforcing, as in Levin (2003).

We denote by $z_t \in Z$ the observable but unverifiable signal of management quality a_t chosen by an old manager in period t. We assume that Z is finite and $z_t \in Z$ is realized with a probability of $p(z_t|a_t) \in (0, 1)$ conditional on the manager's choice of management quality a_t. Given this, each old entrepreneur offers a wage scheme, defined as the mapping $b_t : Z \to \mathbb{R}_+$, to an old manager, where b_t specifies a non-negative wage contingent on the realization of the signal $z_t \in Z$. As before, the old entrepreneur asks a widget producer to produce k_t units of widgets and pays the widget producer the corresponding cost of k_t.[25]

The total payments to the old manager and widget producer $k_t + b_t(z)$ cannot exceed the firm value of V_t minus the liquidation payoff L, as follows

$$V_t - L \geq k_t + b_t(z), \quad \text{for any } z \in Z \tag{DE*}$$

This is because the old entrepreneur can always renege on k_t and $b_t(z)$ after observing the realization of the observable signal z in period t, and then liquidate the firm's production asset. By a similar argument to Lemma 2, we can show that DE* becomes the necessary and sufficient condition for the old entrepreneur to honor the agreed upon payments k_t and $b_t(z)$ for any realization of the unverifiable signal $z \in Z$.

Given DE*, the old manager chooses a management quality level of a_t when his IC constraint is satisfied:

$$\sum_z p(z|a_t)b_t(z) - c(a_t) \geq \sum_z p(z|a)b_t(z) - c(a) \quad \text{for any } a \neq a_t. \tag{IC}$$

In addition, each old manager accepts the offered contract $\{a_t, b_t\}$ when his IR constraint is satisfied:

$$\sum_z p(z|a_t)b_t(z) - c(a_t) \geq 0 \tag{IR}$$

Finally, the limited liability (LL) constraint must hold:

$$b_t(z) \geq 0, \quad \text{for any } z \in Z \tag{LL}$$

[25] More generally, the widget producer's payment can also be contingent on the signal realization $z \in Z$. We do not pursue this complicated case here. One justification for this is that widget producers need to consume before the signal realization and must receive non-contingent payments before $z \in Z$ is realized. Appendix C provides a more general case that allows the payments to widget producers to be contingent on z.

Each old entrepreneur chooses a relational incentive contract $\{a_t, b_t, k_t\}$ to maximize her expected profit,

$$F(h(a_t, A_{t-1})) - \sum_z p(z|a_t)b_t(z) - k_t$$

subject to IC, IR, LL, and DE*, given V_t and A_{t-1}.

We can then show that our main results, Propositions 1 and 2, still remain valid even when we consider the case in which management quality is the old manager's private information (see Appendix B for the details of this result). One implicit assumption behind this extension is that widget producers are paid their costs to produce widget k_t, no matter what signals $z \in Z$ are realized about the old manager's choice of management quality. A more general scheme for widget producers would allow the payments to depend on the signal realization; that is, $R_t(z)$ for each $z \in Z$. Then, we modify the DE constraint to $V_t - L \geq R_t(z) + b_t(z)$ for all $z \in Z$, which is weaker than the original constraint DE*. This makes the set of feasible relational contracts larger, so that old entrepreneurs can increase their profits. In Appendix C, we treat this general case separately and show that equilibrium paths still become cyclical.

7 Conclusion

This study presented a dynamic theory of management cycles to account for how firms switch between management regimes over time. Firms accumulate managerial capital, which are intangible assets such as management know-how and knowledge, by motivating the managers to acquire management qualities that contribute to the accumulation of managerial capital. This affects and is affected by the DE constraint, that is, the extent to how much the firm can credibly commit to pay to trading parties, the manager, and the widget producer. The firm's commitment capacity depends on its market value, which reflects the streams of its future profits. We then showed that the interactions between the DE constraint and the accumulation of managerial capital causes endogenous cycles of management quality over time. That is, firms alternate between more and less productive management quality as profits fluctuate over time, which helps clarify how world-leading companies often lose managerial fidelity through corporate scandals and how they recover from such management failures.

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8 Appendix A: Proofs

Proof of Lemma 1.
A^i is the steady-state level of managerial capital that satisfies $A^i = h(a^i, A^i)$ for $a^i \in \mathbb{A}$. A^i depends on the parameter $\delta \in [0,1]$ implicitly. Since h is concave with respect to A, we obtain $h(a^i, A^i) - h(a^i, 0) \geq \eta(a^i, A^i, \delta)A^i$, which implies that $A^i \geq h(a^i, 0)/(1-\eta(a^i, A^i; \delta))$ according to the definitions of A^i and $\eta(a, A; \delta)$ and that $\eta(a^i, A^i; \delta) < 1$ (h must cross the 45 degree line from above at the steady state.) Suppose that $\lim_{\delta \to 1} A^i \equiv \hat{A} < +\infty$. Then, if we take δ to be close to 1, we can ensure that $\eta(a^i, \hat{A}; \delta) \to 1$ for a finite $\hat{A}$. However, this implies that $\lim_{\delta \to 1} A^i = \hat{A} \geq \lim_{\delta \to 1} h(a^i, 0)/(1-\eta(a^i, \hat{A}; \delta)) = \infty$, which is a contradiction. Thus, $\hat{A} = \infty$ must hold when δ is close to 1. Q.E.D.

Proof of Lemma 2.
Necessity: Shown in the main text.
Sufficiency: Consider an old entrepreneur, an old manager, and a widget producer in firm i, who agree on an informal contract $\{R_t, k_t, b_t, a_t\}$ at the beginning of period t. Suppose that DE is satisfied. Then, suppose that the old entrepreneur reneges on the agreed payments of R_t or b_t in period t after the widget producer and the old manager choose k_t and a_t according to the informal contract. The young manager employed by the firm in period t observes the old entrepreneur's deviation.

Then, we consider the following strategies by the young managers and the old entrepreneurs, who are matched in firm i in future periods: (i) The young manager hired by firm i in period $s \geq t$ does not exercise his separation option (he does not leave firm i) if the old manager (who was the young manager in the previous period, $s-1$) remains in firm i in period s, the old manager and widget producer choose the agreed upon actions a_s and k_s, and then receive the agreed upon payments of b_s and R_s from the old entrepreneur in period s. Otherwise, the young manager leaves firm i in period s. (ii) The old entrepreneur who owns firm i in period $s \geq t$ does not exercise her separation option (she does not liquidate firm i) if the old manager (who was the young manager in period $s-1$) remains in the firm in period s, the old manager and the widget producer choose the agreed upon actions a_s and k_s, and receive the agreed upon payments of b_s and R_s in period s. Otherwise, the old entrepreneur liquidates firm i's production assets in period s.

We show that the strategies above constitute a continuation equilibrium after the old entrepreneur's deviation in period t. First, given the liquidation decision by the old entrepreneur in period t, it is a best response for the young manager hired by firm i in period t to leave the firm because he obtains the same payoff, zero, whatever he does. Second, given the young manager's decision to leave in period t, the old entrepreneur would obtain $y_t + L$ by exercising her liquidation option. To see that this can be a best response for the old entrepreneur, suppose that she does not liquidate the firm, given the young manager's decision to leave the firm in period t. Then, following the strategies specified above, the young manager and the old entrepreneur, who will be new members of firm i in the next period, $t+1$, will exercise their separation options simultaneously, resulting in the liquidation of the firm in period $t+1$. Then, from DE and the liquidation in period $t+1$, it must be that $R_{t+1} = 0$, and hence $k_{t+1} = 0$, which in turn implies that $\pi_{t+1} = 0$ holds because $F(A_t, 0) = 0$. From the no-arbitrage condition $rV_t = \pi_{t+1} + L$ in period t (noting that the firm will be liquidated in period $t+1$, and hence L will be obtained), $V_t = L/r$ follows so that $V_t = L/r < L$. Hence the old entrepreneur in period t will obtain $y_t + L$ by liquidating the firm after her deviation in period t whereas she will obtain $y_t + V_t$ by not liquidating the firm. Since the former is larger than the latter, it is a best response for the old entrepreneur to liquidate the firm given the young manager's quitting decision in period t after the old entrepreneur's deviation in period t.

Thus, we establish the existence of a continuation equilibrium in which the young manager and the old entrepreneur exercise their separation options simultaneously after the latter's deviation by reneging on b_t or R_t at the end of period t. Q.E.D.

Proof of Lemma 3. Note that

$$\max_{k:V-L-c(a)\geq k} F(h(a,A^0),k) - k - c(a) \geq F(h(a,A^0), V-L-c(a)) - (V-L)$$

because each old entrepreneur can always choose $k = V - L - c(a)$. Then, Assumption 4 implies that $G(V;a) > 0$ holds for some $V \geq L + c(a)$ and some $a \neq a^0$ because

$$G(V,a) \geq F(h(a,A^0), V-L-c(a)) - (V-L) - (r-1)V > 0.$$

Q.E.D.

Proof of Proposition 1.
The proof of Proposition 1 consists of two parts. First, we show that no equilibrium paths that are convergent in any "good" management quality $a^i \neq a^0$ exist, provided $\delta \in [0,1]$ is sufficiently close to 1 (Proposition A1). Second, we also show that no equilibrium paths that are convergent in the lowest management quality a^0 exist provided a^0 is close to zero (Proposition A2). By combining these two results, we complete the proof of Proposition 1.

Proposition A1. *Suppose that δ is sufficiently close to 1. Then, there exist no equilibrium paths that are convergent in any higher management quality a^i than the lowest one a^0, that is, $a^i \neq a^0$.*

Proof. Suppose the existence of an equilibrium path that is convergent in some management quality $a^i \neq a^0$. That is, $a_t = a^i$ holds for all $t \geq T$ from some period T onward. Then, we have the limit $A_t \to A^i = h(a^i, A^i)$ as $t \to \infty$. The equilibrium profit in period $t \geq T$ is

$$\tilde{\Phi}(a^i; V_t, A_{t-1}) \equiv \max_{k: V_t - L - c(a^i) \geq k} F(h(a^i, A_{t-1}), k) - k - c(a^i)$$

when $a_t = a^i$ is implemented. Additionally, the market value of the firm in period $t \geq T$ is

$$rV_{t-1} = \tilde{\Phi}(a^i; V_t, A_{t-1}) + V_t.$$

When $t \to \infty$, A_{t-1} converges to A^i: for any $\varepsilon > 0$, there exists some $t^* > T$ such that for all $t \geq t^*$,

$$|A_{t-1} - A^i| < \varepsilon.$$

Thus, we have

$$rV_{t-1} \geq \tilde{\Phi}(a^i; V_t, A^i - \varepsilon) + V_t$$

and

$$rV_{t-1} \leq \tilde{\Phi}(a^i; V_t, A^i + \varepsilon) + V_t$$

for all $t \geq t^* > T$ because $\tilde{\Phi}$ is increasing in A_{t-1}. When δ is close to 1, $A^i \to \infty$ due to Lemma 1. Thus, there exists the smallest value of V, denoted by $\underline{V}(A^i)$, such that $rV = \Phi(V, A^i) + V$ and $\underline{V}(A^i) > 0$. We also have $\underline{V}(A^i + \varepsilon) > 0$ and $\underline{V}(A^i - \varepsilon) > 0$ for a small $\varepsilon > 0$ because Φ is continuous in V and A. Note here that $\underline{V}(A^i - \varepsilon) > \underline{V}(A^i + \varepsilon)$. If some $V \neq \underline{V}(A^i)$ exists such that $rV = \Phi(V, A^i) + V$, then such value V diverges to infinity when $\delta \to 1$ (thus, $A^i \to \infty$, and hence $\Phi(V, A^i) \to \infty$).

Moreover, since $\Phi(c(a^i) + L, A^i) = 0$ holds for all $A^i \geq 0$, $\underline{V}(A^i) > c(a^i) + L$ is satisfied for all large $A^i > 0$. In addition, we have $\underline{V}(A^i) \to c(a^i) + L$ as $\delta \to 1$. We prove this last point as follows. When $\delta \to 1$, we have $A^i \to \infty$, in which case $\Phi(V, A^i) \to \infty$. Since $\Phi(V, A^i) - (r-1)V$ becomes strictly negative at $V = L + c(a^i)$ for any $\delta \in (0, 1)$, and takes a large positive value when $\delta \to 1$, $\underline{V}(A^i)$ (the value of V such that $\Phi(V, A^i) - (r-1)V = 0$) approaches $L + c(a^i)$ as $\delta \to 1$.

Then, we show that the firm's equilibrium market value V_t must eventually converge to $\underline{V}(A^i)$. Note that $\underline{V}(A^i)$ is the only value of V that can satisfy both $rV = \Phi(V, A^i) + V$ and $V \leq I$ when $\delta \to 1$.[26] Thus, $(1/r)\{\Phi(V, A^i) + V\} \geq V$ for all $V \in [\underline{V}(A^i), I]$ and $(1/r)\{\Phi(V, A^i) + V\} \leq V$ for all $V \in [c(a^i) + L, \underline{V}(A^i)]$. First, as long as $V_{t-1} \in [\underline{V}(A^i - \varepsilon), I]$, we have $(1/r)\{\Phi(V_{t-1}, A^i - \varepsilon) + V_{t-1}\} \geq V_{t-1}$, which imply that $\Phi(V_t, A^i - \varepsilon) \geq (r-1)V_{t-1} > 0$, and hence $\Phi(V_t, A^i - \varepsilon) > 0$. Thus, $\Phi(V_t, A_{t-1}) > \Phi(V_t, A^i - \varepsilon)$. This implies that

$$\begin{aligned} \Phi(V_{t-1}, A^i - \varepsilon) + V_{t-1} &\geq rV_{t-1} \\ &= \Phi(V_t, A_{t-1}) + V_t \\ &> \Phi(V_t, A^i - \varepsilon) + V_t \end{aligned}$$

[26] Any other values of V satisfying $rV = \Phi(V, A^i) + V$ diverges to infinity as $\delta \to 1$, as we have already shown.

showing that $V_{t-1} > V_t$ because Φ is non-decreasing in V. Thus, V_t declines over time as long as $V_t \geq \underline{V}(A^i - \varepsilon)$. Similarly, when $V_{t-1} \in L + c(a^i), \underline{V}(A^i + \varepsilon)]$, we have $V_{t-1} \geq (1/r)\{\Phi(V_{t-1}, A^i + \varepsilon) + V_{t-1}\}$ such that

$$\begin{aligned} \Phi(V_{t-1}, A^i + \varepsilon) + V_{t-1} &\leq rV_{t-1} \\ &= \Phi(V_t, A_{t-1}) + V_t \\ &\leq \Phi(V_t, A^i + \varepsilon) + V_t \end{aligned}$$

which shows that $V_t \geq V_{t-1}$. Here, we can show that the strict inequality $V_t > V_{t-1}$ holds. If $\Phi(V_t, A_{t-1}) = 0$, then $rV_{t-1} = V_t$ holds so that $V_t > V_{t-1}$ due to $r > 1$ and $V_{t-1} \geq L > 0$. If $\Phi(V_t, A_{t-1}) > 0$, then $\Phi(V_t, A_{t-1}) < \Phi(V_t, A^i + \varepsilon)$ so that the second inequality in the above expressions is strict, and hence $V_t > V_{t-1}$ as well. Thus, V_t increases over time as long as $V_t \leq \underline{V}(A^i + \varepsilon)$.

These arguments establish that $V_t \in (\underline{V}(A^i + \varepsilon), \underline{V}(A^i - \varepsilon))$ eventually holds as $t \to \infty$. By taking $\varepsilon > 0$ to be sufficiently small, we obtain the limit $V_t \to \underline{V}(A^i)$ as $t \to \infty$.

We now consider the deviation by an old entrepreneur who implements the lowest management quality a^0 instead of the equilibrium quality $a^i \neq a^0$ in some period $t \geq t^*$. In so doing, the old entrepreneur offers the old manager a wage contract of $b_t = c(a^0)$ implementing the lowest quality a^0 and choosing the widget production of $k_t = V_t - L - c(a^0)$. Here, since $\underline{V}(A^i) > c(a^i) + L$ and $c(a^i) > c(a^0)$, DE is satisfied. That is, $V_t > c(a^0) + L$ and hence $k_t = V_t - L - c(a^0) > 0$. Then, the deviation profit can be at least

$$F(h(a^0, A_{t-1}), V_t - L - c(a^0)) - (V_t - L) \geq F(h(a^0, A^i - \varepsilon), c(a^i) - c(a^0)) - (V_t - L),$$

where $V_t \geq c(a^i) + L$.

On the other hand, the equilibrium profit is bounded above by

$$\Phi(V_t, A_{t-1}) = rV_{t-1} - V_t \leq r\underline{V}(A^i - \varepsilon) - \underline{V}(A^i + \varepsilon)$$

because $V_t \in (\underline{V}(A^i + \varepsilon), \underline{V}(A^i - \varepsilon))$ for all large t. Letting $\varepsilon \to 0$ as $t \to \infty$, the upper bound for the equilibrium profit, which we defined above, is close to $(r-1)\underline{V}(A^i)$, which approaches $(r-1)(c(a^i) + L)$ when $\delta \to 1$ because $\underline{V}(A^i) \to c(a^i) + L$ when $\delta \to 1$. Thus, if δ is so close to 1 that $A^i \to \infty$, then we have

$$F\left(h(a^0, A^i), c(a^i) - c(a^0)\right) + L > r\underline{V}(A^i) \simeq r(c(a^i) + L)$$

for any $a^i \neq a^0$; then, the deviation above becomes profitable. Q.E.D.

Next, we show that no equilibrium that is convergent in the lowest management quality a^0 exists.

Proposition A2. *Suppose that the lowest management quality a^0 has only a small contribution to the accumulation of managerial capital; that is, $a^0 \simeq 0$. Then, no equilibrium paths that are convergent in the lowest management quality a^0 exist.*

Proof. Suppose the existence of an equilibrium path that is convergent in the lowest management quality. That is, $a_t = a^0$ holds for all $t \geq T$ from some period T onward. Thus, we have the limit $A_t \to A^0 = h(a^0, A^0)$ as $t \to \infty$. Then, the firm's market value evolves according to

$$rV_{t-1} = \tilde{\Phi}(a^0; V_t, A_{t-1}) + V_t$$

from period T onward, where A_{t-1} is close to A^0, and

$$\tilde{\Phi}(a^0; V_t, A_{t-1}) = \max_{k: V_t - L - c(a^0) \geq k \geq 0} F(h(a^0, A_{t-1}), k) - k - c(a^0).$$

When $A_{t-1} \to A^0$ and a^0 is small enough ($a^0 \to 0$), the above profit $\tilde{\Phi}(a^0, V_t, A_{t-1})$ becomes close to zero because

$$\max_k F(h(a_0, A_{t-1}), k) - k - c(a^0) \simeq -c(a^0) \leq 0$$

when $A_{t-1} \simeq A^0 \simeq 0$. Then, we define $\tilde{V}^0$ as the lowest value of V satisfying $r\tilde{V}^0 = \tilde{\Phi}(a^0, \tilde{V}^0, A^0) + \tilde{V}^0$ if it exists. If it does not exist, we define $\tilde{V}^0 = +\infty$, which is the case when $(1/r)\{\tilde{\Phi}(a^0; V, A^0) + V\} < V$ for all $V \geq 0$. By the definition of $\tilde{V}^0$, $V_t \geq \tilde{V}^0 - \varepsilon$ must hold for all large t for any given small $\varepsilon > 0$. To see this, note that we have $(1/r)\{\tilde{\Phi}(a^0, V_t, A_{t-1}) + V_t\} \simeq (1/r)\{\tilde{\Phi}(a^0, V_t, A^0) + V_t\} < V_t$ for all $V_t < \tilde{V}^0$ when $t \to \infty$ so that $A_{t-1} \to A^0$. Thus, $V_{t+1} > V_t$ holds as long as $V_t < \tilde{V}^0$. Therefore, V_t increases over time, which implies that $V_t \geq \tilde{V}^0 - \varepsilon$ for all large t for any given small $\varepsilon > 0$.

$\tilde{V}^0 = +\infty$ holds when a^0 is so small. Thus, V_t becomes so large that $V_t > L + \lambda h(a^*(A_{t-1}), A_{t-1}) + c(a^*(A_{t-1}))$ eventually holds for any large enough t when a^0 is small enough. Here, note that A_t is bounded above by $A_{\max} \equiv \max\{A^m, A_0\}$, which implies that $h(a^*(A_{t-1}), A_{t-1}) \leq h(a^m, A_{\max})$. Thus, DE must be slack for any large enough t. Given this, if an old entrepreneur deviated to implement $a^*(A_{t-1})$ instead of the equilibrium a^0, she would earn a strictly positive profit of $\psi h(a^*(A_{t-1}), A_{t-1}) - c(a^*(A_{t-1})) > 0$ under Assumption 3. Since her equilibrium profit $\tilde{\Phi}(a^0; V_t, A_{t-1})$ becomes close to zero when $a^0 \to 0$, the old entrepreneur can make a profitable deviation. Q.E.D.

We now prove Proposition 1. We fix a $\delta \in (\overline{\delta}, 1)$ such that for all $a^i \neq a^0$,

$$F\left(h(a^0, A^i), c(a^i) - c(a^0)\right) + L > r(\underline{V}(A^i) - L)$$

and the lowest management quality as $a^0 \in (0, \overline{a}^0)$ such that

$$\tilde{V}^0 > \max_{0 \leq A \leq A_{\max}} \lambda h(a^*(A), A) + c(a^*(A)) + L.$$

Then, we know from Propositions A1 and A2 that there are no equilibrium paths that are convergent in any good management $a^i \neq a^0$, and the lowest management a^0. Thus, every equilibrium must be cyclical, completing the proof of Proposition 1. Q.E.D.

Proof of Proposition 2.
We first show the following claim.

Lemma A1. *Suppose that* $A_0 \in (A^0, A^m)$. *Then,* $V_t \in (\underline{V}^m, \overline{V}^0)$ *holds for all* $t \geq 0$ *when* $V_0 \in (\underline{V}^m, \overline{V}^0)$.

Proof. Suppose that $A_0 \in (A^0, A^m)$ and $V_0 \in (\underline{V}^m, \overline{V}^0)$. Note that $A_t \in (A^0, A^m)$ holds for all $t \geq 0$.

Since $A_{t-1} \leq A^m$ for all t, we have

$$\begin{aligned} rV_{t-1} &= \Phi(V_t, A_{t-1}) + V_t \\ &\leq \Phi(V_t, A^m) + V_t. \end{aligned}$$

When $V_{t-1} \geq \underline{V}^m$, we obtain

$$\begin{aligned} \Phi(\underline{V}^m, A^m) + \underline{V}^m &= r\underline{V}^m \\ &\leq rV_{t-1} \\ &\leq \Phi(V_t, A^m) + V_t \end{aligned}$$

which shows that $V_t \geq \underline{V}^m$ because Φ is non-decreasing in V. Similarly, when $V_{t-1} \leq \overline{V}^0$, we obtain

$$\begin{aligned} \Phi(\overline{V}^0, A^0) + \overline{V}^0 &= r\overline{V}^0 \\ &\geq rV_{t-1} \\ &= \Phi(V_t, A_{t-1}) + V_t \\ &\geq \Phi(V_t, A^0) + V_t \end{aligned}$$

which shows that $\overline{V}^0 \geq V_t$ (the shaded area in Figure A1 depicts the region of equilibrium firm market value $(\underline{V}^m, \overline{V}^0)$ using the definitions of $\overline{V}^0$ and $\underline{V}^m$.)

Thus, $V_t \in (\underline{V}^m, \overline{V}^0)$ holds as long as $V_{t-1} \in (\underline{V}^m, \overline{V}^0)$. Q.E.D.

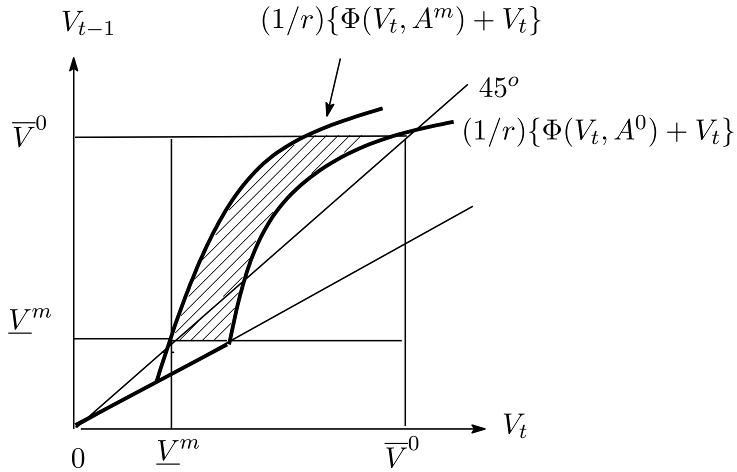

Figure A1: Region of equilibirum values

Next, we show the following.

Lemma A2. *Suppose that $\delta \to 1$ and $a^0 \simeq 0$. Then, every equilibrium path $\{A_t, V_t\}_{t=0}^{\infty}$ that satisfies $V_0 \in (\underline{V}^m, \overline{V}^0)$ and $A_0 \in (A^0, A^m)$ must be cyclical.*

Proof. Take any $a^i \neq a^0$. First, note that some $a^i \neq a^0$ exists such that $c(a^i) + L < \overline{V}^0$. To see this, take some $V < \overline{V}^0$. Then, when V is close to $\overline{V}^0$, Assumption 3 implies that the following value is strictly positive:

$$\max_{a \neq a^0, k} F(h(a, A^0), k) - k - c(a)$$

subject to $V - k \geq L + c(a)$. Thus, firms can earn a positive profit when V is close to $\overline{V}^0$. Letting the optimal choices of k and a in the above problem be $\hat{k}$ and $\hat{a} \neq a^0$, respectively, then $V - \hat{k} \geq L + c(\hat{a})$ so that $V > c(\hat{a}) + L$. This shows that $\overline{V}^0 > V > L + c(\hat{a})$ and hence $\overline{V}^0 > c(\hat{a}) + L$ for such $\hat{a} \neq a^0$.

Now, suppose that $\delta \in (0, 1)$ is so large ($A^i \to \infty$) that for all $V \in [c(a^i) + L, \overline{V}^0]$ and all $a^i \neq a^0$ such that $c(a^i) + L < \overline{V}^0$, we have

$$F\left(h(a^0, A^i), V - L - c(a^0)\right) + L > rV. \quad (*)$$

Here, by the above argument, we can find some $a^i \neq a^0$ such that $\overline{V}^0 > c(a^i) + L$.

Take any equilibrium path $\{A_t, V_t\}$ such that $A_t \in (A^0, A^m)$ and $V_t \in (\underline{V}^m, \overline{V}^0)$ hold for all $t \geq 0$. If $a_t = a^i$ holds for some $a^i \neq a^0$ in any period $t \geq T$ from period T onward, then $A_t \to A^i$ as $t \to \infty$. Additionally, $V_t \geq L + c(a^i)$ must hold for all $t \geq T$ for such a path to be an equilibrium because otherwise, $k_t = 0$ follows. (i) Let $V_t \geq V_{t-1}$ in some period $t \geq T$.

Then, the equilibrium profit of an old entrepreneur is

$$\Phi(V_t, A_{t-1}) = \max_{k_t} \; F(h(a^i, A_{t-1}), k_t) - k_t - c(a^i)$$

subject to $V_t - L - c(a^i) \geq k_t \geq 0$ in period $t \geq T$. Since $rV_{t-1} = \Phi(V_t, A_{t-1}) + V_t$ and $V_t \geq V_{t-1}$, we have

$$\Phi(V_t, A_{t-1}) = rV_{t-1} - V_t \leq (r-1)V_t.$$

However, if an old entrepreneur deviates to implement $a_t = a^0$ and chooses $k_t = V_t - L - c(a^0)$ in period t, she can earn at least

$$F(h(a^0, A_{t-1}), V_t - L - c(a^0))) - (V_t - L)$$

where $V_t - L \geq c(a^i) > c(a^0)$. By the above condition (*), this deviation profit can be greater than $(r-1)V_t$ because $A_{t-1} \to A^i$. Thus, the deviation becomes profitable. (ii) Let $V_{t-1} < V_t$ for all $t \geq T$. Then, since $V_t \geq c(a^i) + L$ must be satisfied in that equilibrium, V_t is a decreasing sequence and is bounded below from $L + c(a^i)$. Thus, such a decreasing sequence $\{V_t\}$ converges to some $\tilde{V}^i \in [c(a^i) + L, \overline{V}^0]$. Then, the equilibrium profit becomes close to

$$\Phi(\tilde{V}^i, A^i) = \max_{k : \tilde{V}^i - L - k \geq c(a^i)} F(h(a^i, A^i), k) - k - c(a^i)$$

and the firm value becomes close to $r\tilde{V}^i = \Phi(\tilde{V}^i, A^i) + \tilde{V}^i$ for a sufficiently large $t > T$. Thus, $\Phi(\tilde{V}^i, A^i) \simeq (r-1)\tilde{V}^i$ holds for large enough t. However, by the deviation to implement $a_t = a^0$ and choose $k = V_t - c(a^0) - L$, the old entrepreneur can earn at least

$$F(h(a^0, A_{t-1}), V_t - L - c(a^0)) - (V_t - L),$$

which is sufficiently close to

$$F(h(a^0, A^i), \tilde{V}^i - L - c(a^0))) - (\tilde{V}^i - L)$$

when $t \to \infty$, implying that $V_t \to \tilde{V}^i$ and $A_{t-1} \to A^i$. This profit can be strictly greater than $(r-1)\tilde{V}^i = \Phi(\tilde{V}^i, A^i)$ under condition (*). Thus, the above deviation becomes profitable.

In either case, any equilibrium path such that $V_t \in (\underline{V}^m, \overline{V}^0)$ for all $t \geq 0$ must not have the property that, for any $a^i \neq a^0$, $a_t = a^i$ for all $t \geq T$ for some period T.

We next show that any equilibrium path such that $V_t \in (\underline{V}^m, \overline{V}^0)$ for all $t \geq 0$ must not have the property that $a_t = a^0$ holds for all $t \geq T$ from period T onward. Suppose that such an equilibrium path exists. Then, the equilibrium profit in $t \geq T$ becomes

$$\max_{k : V_t - k_t - L \geq c(a^0)} F(h(a^0, A_{t-1}), k_t) - k_t - c(a^0).$$

Since $A_{t-1} \to A^0$ as $t \to \infty$ and $A^0 \simeq 0$ for $a^0 \simeq 0$, the above profit tends to be zero, so $\tilde{\Phi}(a^0; V_t, A_{t-1}) = 0$. Thus, the firm's market value must satisfy $rV_{t-1} = V_t$ for all large enough t. This implies that $\lim_{t\to\infty} V_t \to \infty$, which is impossible because $V_t \in [\underline{V}^m, \overline{V}^0]$ for all t.

We have thus established that any path of $\{A_t, V_t\}_{t=0}^{\infty}$ that satisfies $V_0 \in (\underline{V}^m, \overline{V}^0)$ and $A_0 \in (A^0, A^m)$ must be cyclical. Q.E.D.

We then use Lemmas A1 and A2 to prove Proposition 2 as follows.

Since V_0 is a jump variable, we can choose any $V_0 \in (\underline{V}^m, \overline{V}^0)$. On the other hand, the initial value of managerial capital A_0 is exogenously given and we assume that it is $A_0 \in (A^0, A^m)$. Then, we can construct an equilibrium path $\{V_t, A_t\}_{t=0}^{\infty}$ that starts from $A_0 \in (A^0, A^m)$ and $V_0 \in (\underline{V}^m, \overline{V}^0)$ and inductively satisfies

$$rV_{t-1} = \Phi(V_t, A_{t-1}) + V_t$$

and

$$A_t = h(\hat{a}(V_t, A_{t-1}), A_{t-1})$$

from $t = 1$ onward.[27] As we showed, such a path satisfies $A_t \in (A^0, A^m)$ for all $t \geq 0$. Further, Lemma A1 shows that $V_t \in (\underline{V}^m, \overline{V}^0)$ holds for all $t \geq 0$ as well. Thus, the path $\{V_t, A_t\}_{t=0}^{\infty}$ never moves out of the region $(A^0, A^m) \times (\underline{V}^m, \overline{V}^0)$, but stays there forever. Lemma A2 then shows that the path $\{V_t, A_t\}_{t=0}^{\infty}$ defined above must be cyclical. Moreover, under Assumption 4 ($\overline{V}^0 < I$), $V_t \leq \overline{V}^0 < I$ so that $V_t < I$ for all $t \geq 0$. Since $V_t \geq \underline{V}^m > c(a^m) + L > L$, we also have $V_t \geq L$ for all $t \geq 0$. Then, $V_t \in [L, I]$ is satisfied for all $t \geq 0$. Thus, the path $\{V_t, A_t\}_{t=0}^{\infty}$ defined above actually becomes an equilibrium path and is cyclical. Q.E.D.

Proof of Proposition 3. We show that in any equilibrium, there exists some $\varepsilon > 0$ such that for any T, some $t \geq T$ exists to ensure that $|\pi_{t+1} - \pi_t| > \varepsilon$. To see this, suppose in contrary to this claim that for any $\varepsilon > 0$ there exists some T such that for any $t \geq T$, $|\pi_{t+1} - \pi_t| \leq \varepsilon$.

We define $\Delta\pi_t \equiv \pi_t - \pi_{t-1}$ and $\Delta V_t \equiv V_t - V_{t-1}$. Then, the no-arbitrage condition (8) in the main text shows that

$$r\Delta V_{t-1} = \Delta\pi_t + \Delta V_t$$

where $-\varepsilon < \Delta\pi_t < \varepsilon$ for any large t. Then, by defining $\overline{\Delta V}_\varepsilon \equiv \varepsilon/(r-1)$ and $\underline{\Delta V}_\varepsilon \equiv -\varepsilon/(r-1)$, ΔV_t must lie in $(\underline{\Delta V}_\varepsilon, \overline{\Delta V}_\varepsilon)$ for any large t. If this is not the case in period $t-1$, say $\Delta V_{t-1} \geq \overline{\Delta V}_\varepsilon$, then it follows from $r\Delta V_{t-1} = \Delta\pi_t + \Delta V_t$ that $r\Delta V_{t-1} < \varepsilon + \Delta V_t$, and

[27] Given A_0 and V_0, we take V_1 that satisfies $rV_0 = \Phi(V_1, A_0) + V_1$. Since $H(V, A) \equiv \Phi(V, A) + V$ is increasing and continuous in V, $H(0) = 0$, and $H(\infty) = \infty$, such V_1 uniquely exists for any given rV_0. Then, we can take $A_1 = h(\hat{a}(V_1, A_0), A_0)$ for A_0 and V_1. Again, this A_1 uniquely exists for any given V_1 and A_0, which determines the value of $h(\hat{a}(V_1, A_0), A_0)$. Given A_1 and V_1, we can take V_2 satisfying $rV_1 = \Phi(V_2, A_1) + V_2$. Then, we take $A_2 = h(\hat{a}(V_2, A_1), A_1)$, and so on.

hence $\Delta V_t > r\overline{\Delta V}_\varepsilon$. Repeating this, $\Delta V_t > r^{\tau-t}\Delta V_t$ for any $\tau > t$ so that $\Delta V_t \to \infty$, which contradicts the fact that ΔV_t must be bounded ($\Delta V_t \leq I$ for all t). Similarly, if ΔV_{t-1} is less than $\underline{\Delta V}_\varepsilon$ in period $t-1$, then $\Delta V_t \to -\infty$, which is impossible because $\Delta V_t \geq -I$ for all t.

Letting $t \to \infty$ and then $\varepsilon \to 0$, we obtain $\Delta V_t \to 0$ because $\underline{\Delta V}_\varepsilon \to 0$ and $\overline{\Delta V}_\varepsilon \to 0$ when $\varepsilon \to 0$. By definition, we have

$$\begin{aligned}\Delta\pi_t &= \Phi(V_t, A_{t-1}) - \Phi(V_{t-1}, A_{t-2}) \\ &= \Phi(\Delta V_t + V_{t-1}, A_{t-1}) - \Phi(V_{t-1}, A_{t-2})\end{aligned}$$

where the left hand side $\Delta\pi_t \to 0$ as $t \to \infty$ by our supposition, while the right hand side converges to $\Phi(V_{t-1}, A_{t-1}) - \Phi(V_{t-1}, A_{t-2})$ as $t \to \infty$ because $\Delta V_t \to 0$. Thus, $\Phi(V_{t-1}, A_{t-1}) - \Phi(V_{t-1}, A_{t-2}) \to 0$ follows as $t \to \infty$. Then, by the continuity of Φ, we obtain $A_{t-1} - A_{t-2} \to 0$ as $t \to \infty$. However, when $A_t \simeq A_{t-1}$ for all large t, it follows from $A_t = h(a_t, A_{t-1})$ that $a_t = a^i$ must hold for some a^i for all large t (otherwise, $a_t \neq a_{t-1}$, but then $A_t - A_{t-1} = h(a_t, A_{t-1}) - h(a_{t-1}, A_{t-2}) \simeq h(a_t, A_{t-1}) - h(a_{t-1}, A_{t-1}) \geq \rho > 0$ for some $\rho > 0$, contracting to the fact that $A_t - A_{t-1} \to 0$). This contradicts Proposition 1. Q.E.D.

Proof of Proposition 4. Take any equilibrium path. We first show that DE becomes always binding from some period t^* onward. Since $A_t \in (A^0, A^m)$, we know that $rV_{t-1} \geq \Phi(V_t, A^0) + V_t$ for all $t \geq 1$. If $I \leq \underline{V}^0$, then $V_t \leq I \leq \underline{V}^0$ holds for all t so that we obtain $V_t \leq \underline{V}^0$ for all t. Thus, we suppose that $I > \underline{V}^0$ in the following. Then, by our assumption that $I < \overline{V}^0$, we have $\underline{V}^0 < I < \overline{V}^0$.

Any equilibrium path of V_t must satisfy $V_t \leq I$. Thus, $rV < \Phi(V, A^0) + V$ holds when $\underline{V}^0 < V < I$. When $V_t \geq \underline{V}^0$, we obtain

$$\begin{aligned}rV_t &\leq \Phi(V_t, A^0) + V_t \\ &< \Phi(V_t, A_{t-1}) + V_t \\ &= rV_{t-1}\end{aligned}$$

where the second strict inequality follows from $\Phi(V, A^0) > 0$ for $V > \underline{V}^0$ and $A_{t-1} > A^0$ so that $\Phi(V_t, A_{t-1}) > \Phi(V_t, A^0) > 0$. Thus, $V_t < V_{t-1}$. Repeating this, $V_t < \underline{V}^0$ for any large enough t. Then, by the definition of $\underline{V}^0$, DE must be binding at $V_t \leq \underline{V}^0$. Thus, there exists some period t^* such that for all $t \geq t^*$, DE becomes binding.

We next note that $Y_t = Y^* \equiv rI + I - L$ when DE is binding in period t because $\pi_t = y_t - (b_t + R_t) = y_t - (V_t - L)$. Hence, $rV_{t-1} = \pi_t + V_t = y_t + L$. From the result above, we know that $Y_t = Y^*$ holds from period t^* onward. Let $\mu_T \equiv (1/T)\sum_{t=1}^{T} Y_t$. Then, we obtain

$$\begin{aligned}
\mathrm{Var}_T(Y) &= (1/T)\sum_{t=1}^{T}|Y_t-\mu_T|^2 \\
&= (1/T)\sum_{t=1}^{T}|Y_t-Y^*+Y^*-\mu_T|^2 \\
&= (1/T)\sum_{t=1}^{T}\{|Y_t-Y^*|+|Y^*-\mu_T|\}^2 \\
&= (1/T)\sum_{t=1}^{T}\{|Y_t-Y^*|^2+2|Y_t-Y^*||Y^*-\mu_T|+|Y^*-\mu_T|^2\} \\
&= (1/T)\sum_{t=1}^{t^*}|Y_t-Y^*|^2+|Y^*-\mu_T|(2/T)\sum_{t=1}^{t^*}|Y_t-Y^*|+|Y^*-\mu_T|^2.
\end{aligned}$$

Here, we take T to be so large that for a small given $\rho > 0$,

$$|Y^*-\mu_T| \leq \sqrt{\rho/T}$$

holds. This is possible because

$$\begin{aligned}
|Y^*-\mu_T| &= \Big|Y^*-(1/T)\sum_{s=1}^{T}Y_s\Big| \\
&= (1/T)\Big|\sum_{s=1}^{T}(Y^*-Y_s)\Big| \\
&= (1/T)\Big|\sum_{s=1}^{t^*}(Y^*-Y_s)\Big|
\end{aligned}$$

using $Y_s = Y^*$ for all $t > t^*$. Therefore, we can take a large T to ensure that

$$|Y^*-\mu_T| = (1/T)\Big|\sum_{s=1}^{t^*}(Y^*-Y_s)\Big| \leq \sqrt{\rho/T}.$$

Then, we can write the expression of $\mathrm{Var}_T(Y)$ above as

$$\mathrm{Var}_T(Y) \leq (1/T)\sum_{t=1}^{t^*}|Y_s-Y^*|^2+(1/T)2\sqrt{\rho/T}\sum_{t=1}^{t^*}|Y_s-Y^*|+\rho/T.$$

Next, we find the lower bound for $\mathrm{Var}_T(\pi)$. By Proposition 3, we know that there are infinitely many periods in which $|\pi_{t+1}-\pi_t| \geq \varepsilon$ for some $\varepsilon > 0$ in any equilibrium path. We denote by $J \equiv \{t_1, t_2, ..., t_\tau\}$ the sequence of periods in which $|\pi_{t_i+1}-\pi_{t_i}| \geq \varepsilon$ for $i = 1, 2, ..., \tau$, where we specify τ below. Take a large $T > t_\tau$. For each t_i, we have either

(i) $|\pi_{t_i} - (1/T)\sum_{s=1}^{T}\pi_s| > \varepsilon/2$ or (ii) $|\pi_{t_i} - (1/T)\sum_{s=1}^{T}\pi_s| \leq \varepsilon/2$. Since $|\pi_{t_i+1} - \pi_{t_i}| \geq \varepsilon$, if case (ii) occurs, we obtain

$$\begin{aligned}
\left|\pi_{t_i+1} - (1/T)\sum_{s=1}^{T}\pi_s\right| &= \left|\pi_{t_i+1} - \pi_{t_i} + \pi_{t_i} - (1/T)\sum_{s=1}^{T}\pi_s\right| \\
&\geq |\pi_{t_i+1} - \pi_{t_i}| - \left|\pi_{t_i} - (1/T)\sum_{s=1}^{T}\pi_s\right| \\
&\geq \varepsilon - \varepsilon/2 \\
&= \varepsilon/2
\end{aligned}$$

Thus, we have $|\pi_{t_i} - (1/T)\sum_{s=1}^{T}\pi_s| \geq \varepsilon/2$ or $|\pi_{t_i+1} - (1/T)\sum_{s=1}^{T}\pi_s| \geq \varepsilon/2$ for each $i = 1, 2, ..., \tau$.

We define

$$D \equiv \left\{ i \in J \;\middle|\; |\pi_{t_i} - (1/T)\sum_{s=1}^{T}\pi_s| \geq \varepsilon/2 \text{ or } |\pi_{t_i+1} - (1/T)\sum_{s=1}^{T}\pi_s| \geq \varepsilon/2 \right\}.$$

Then, we can derive the lower bound for $\mathrm{Var}_T(\pi)$:

$$\begin{aligned}
\mathrm{Var}_T(\pi) &= (1/T)\sum_{t=1}^{T}\left|\pi_t - (1/T)\sum_{s=1}^{T}\pi_s\right|^2 \\
&\geq (1/T)\sum_{i\in D}\left\{\left|\pi_{t_i} - (1/T)\sum_{s=1}^{T}\pi_s\right|^2 + \left|\pi_{t_i+1} + (1/T)\sum_{s}\pi_s\right|^2\right\} \\
&\geq (1/T)\tau\varepsilon^2/4
\end{aligned}$$

where $\#D = \tau$ due to the result above.

We now take a large τ such that

$$\tau\varepsilon^2/4 > \sum_{t=1}^{t^*}|Y_s - Y^*|^2 + 2\sqrt{\rho}\sum_{t=1}^{t^*}|Y_s - Y^*| + \rho.$$

Additionally, we fix some $T > \tau$ for this τ. Then, by combining the above upper bound for $\mathrm{Var}_T(Y)$ with the lower bound for $\mathrm{Var}_T(\pi)$, we can show that

$$\begin{aligned}
\mathrm{Var}_T(\pi) &\geq (1/T)\tau\varepsilon^2/4 \\
&> (1/T)\sum_{t=1}^{t^*}|Y_s - Y^*|^2 + (1/T)2\sqrt{\rho/T}\sum_{t=1}^{t^*}|Y_s - Y^*| + \rho/T \\
&\geq \mathrm{Var}_T(Y).
\end{aligned}$$

Q.E.D.

Online Appendix (Not For Publication)

In this appendix we discuss several extensions of the model presented in the main text.

9 Appendix A: Strict Incentives to Exercise the Separation Options

In the main text, we focus on the continuation equilibrium after the old entrepreneur's deviations in which both the old entrepreneur and the young manager exercise their separation options simultaneously. When observing that the young manager left the firm, young entrepreneurs in the same period expect that the firm's market values will be below the liquidation value of L in future periods. This punishes the old entrepreneurs who renege on wages for old managers and payments for widget producers.

However, in this continuation equilibrium, the young manager is indifferent between leaving and not leaving the firm, given the old entrepreneur's liquidation decision. Here, we modify the basic model to derive the continuation equilibrium in which the young manager has a strict incentive to take the separation option.

We consider the following alternative scenario. Each young manager obtains an outside job opportunity in the end of the period, which will pay the young manager $u_y > 0$ if he immediately accepts this outside job. When young managers do not leave and turn the outside job down, they will obtain no payoffs. On the other hand, we assume that young managers can acquire some knowledge about management as long as the firms (production assets) continue to operate without liquidation until the next period. Then, when a young manager becomes old and leaves the firm, he can utilize his acquired management knowhow and obtain a positive outside payoff of $u_o > 0$ by running some side-business himself. Here, we assume that $u_o > u_y > 0$, meaning that managers can obtain a higher outside payoff when they learn management knowledge from actively operating firms than when they immediately leave firms.

Then, each young manager has the strict incentive to quit the firm given the old entrepreneur's liquidation choice because doing so gives him $u_y > 0$ immediately, but he earns nothing by remaining with the firm. On the other hand, when the old entrepreneur does not liquidate the firm in period t, the young manager will obtain the outside payoff of $u_o > 0$ when he becomes old in period $t + 1$ by staying with the firm holding a non-liquidated production asset. However, the young manager earns the outside payoff u_y by leaving immediately at the end of period t. Since $u_o > u_y$, the former is more profitable for the young manager than the latter, provided that the old entrepreneur will not liquidate the firm.

Thus, there exists a strict Nash equilibrium in which a young manager quits the firm and an old entrepreneur liquidates the firm simultaneously with strict incentives after the widget producer or old manager do not receive their agreed upon payments of b_t and R_t, respectively.

10 Appendix B: Unobservable Managerial Choices and Relational Incentive Pay

In this appendix, we investigate the case in which the old manager's choice of management quality level a_t is his own private information. Instead, there is a performance signal, denoted by $z_t \in Z$ about the management quality the old manager has chosen in period t. This signal is observable to all parties within the firm in any given period, but not verifiable to outside parties, and even for future members joining the same firm. Thus, the signal realization $z \in Z$ cannot be contractible. We assume that Z is a finite set and the probability of a signal z_t being realized, denoted by $p(z_t|a_t) \in (0,1)$, depends on the old manager's management quality in period t. Here, $\sum_{z \in Z} p(z|a) = 1$ for all $a \in \mathbb{A}$.

10.1 Self-Enforcing Contracts

We continue to assume that the production level of widget k_t is observable to all relevant parties within a firm in any given period, but not verifiable to outside parties. Furthermore, we assume that only the old entrepreneur who owns the firm in period t can observe the produced output y_t and the managerial capital A_t. Thus, y_t and A_t cannot be a part of the informal contract agreements.[28]

Given all of the assumptions above, a wage scheme offered to an old manager can be contingent only on the signal $z \in Z$, which all parties in a firm can commonly observe in period t. We define such a wage scheme as the mapping $b_t : Z \to \mathbb{R}_+$, which specifies a wage $b_t(z)$ to an old manager for each realization of his performance signal $z \in Z$. Since managers have no wealth, their wages must be non-negative, $b_t(z) \geq 0$ for all $z \in Z$. This wage scheme must be self-enforced by the old entrepreneur because the performance signal z_t is not verifiable.

We make the following weak restriction on the probability distribution of the performance signal $z \in Z$.

[28]When only the old entrepreneur can observe both y_t and A_t, she will not reveal the information about them truthfully: even when the wage scheme b_t for an old manager depends on the old entrepreneur's report on y_t and A_t, she will always minimize b_t in reporting on y_t and A_t. Here, we rule out the scheme to "burn money," such that the old entrepreneur is allowed to discard resources ex-post. If we allow the money burning scheme, the old entrepreneur may tell the truth, but we do not consider such a scheme because it it not ex-post efficient.

Assumption B1. *There exist no m-dimensional non-negative vectors $\lambda \in \mathbb{R}^m$, in which no elements are zero, such that for any management quality $a^i \neq a^0$,*

$$p(z|a^i) = \sum_{k \neq i} \lambda_k p(z|a^k), \quad \textit{for each } z \in Z.$$

Assumption B1 states that for any management quality $a^i \neq a^0$, which is not the least costly one, the old manager cannot induce the same probability distribution of the performance signals as in a^i by choosing other mixed strategies over management qualities. Assumption B1 is generically satisfied for any probability distribution when the number of signals $z \in Z$ is larger than that of possible management quality $a \in \mathbb{A}$; that is, when $\#Z \geq \#\mathbb{A}$. The previous studies about incentive contracts often use similar conditions.[29]

A wage contract offered to an old manager $\{b_t, a_t\}$ must satisfy the following three constraints:

$$E[b_t(z_t)|a_t] - c(a_t) \geq 0 \tag{IR-M}$$

$$E[b_t(z_t)|a_t] - c(a_t) \geq E[b_t(z_t)|a''] - c(a'') \quad \text{for any } a'' \neq a_t \tag{IC}$$

$$b_t(z) \geq 0 \tag{LL}$$

IR-M refers to the old manager's individual rationality constraint to accept the offered contract instead of rejecting it and obtaining his reservation payoff of zero. IC refers to the incentive compatibility constraint for the old manager to optimally choose the management quality of a_t that the entrepreneur wants to implement. LL refers to the limited liability constraint that the old manager's wages must be non-negative for any realized signal $z \in Z$. Since there are no moral hazard problems for young managers, they receive a wage of zero.

Additionally, as in the basic model, the old entrepreneur makes an informal contract to pay k_t to a widget producer in exchange for producing k_t units of widgets in period t.[30]

10.2 Dynamic Enforcement

Since both widget production k_t and the signal realization z_t are non-verifiable, the old entrepreneur may breach the informal agreements to pay k_t to the widget producer and wage $b_t(z)$ to the old manager. In fact, after the signal z_t is realized, the old entrepreneur can obtain a payoff of at least $y_t + L$ by refusing these payments and liquidating the firm's production asset. On the other hand, if she followed the agreement to pay k_t and $b_t(z)$ and then sold her ownership of the firm at the market price of V_t, she would obtain an equilibrium

[29] See, for example, Hermalin and Katz (1991).

[30] Here, we assume that the payment to the widget producer is not contingent on the performance signal $z \in Z$ of the old manager's chosen management quality. We will extend the model to allow such contingent payments to widget producers in Appendix C below.

payoff of $y_t - (R_t + b_t(z)) + V_t$. Thus, the old entrepreneur self-enforces the agreed upon contract only if the former payoff is not greater than the latter payoff; that is, the following *dynamic enforcement* (DE-B) constraint is satisfied:

$$V_t - L \geq k_t + b_t(z), \quad \text{for any } z \in Z. \tag{DE-B}$$

DE-B is a modified version of the DE constraint we considered in the main text.

Conversely, using a similar logic to Lemma 2, we can show that DE-B is sufficient for the old entrepreneur to honor payments under the informal agreement k_t and $b_t(z)$.

10.3 Optimal Contracts

We begin by solving the problem to minimize the expected wages paid to an old manager to implement a management quality level a_t given V_t, A_{t-1} and k_t. By letting $\xi \equiv V - L - k$ in DE-B, we have

$$\xi \geq b_t(z), \tag{DE-B}$$

and the old entrepreneur solves the following minimization problem:

Problem M

$$\min_{b_t} \sum_{z \in Z} p(z|a_t) b_t(z)$$

subject to IR-M, IC, LL, and DE-B.

We denote by $\Gamma(\xi; a)$ the constraint set in Problem M and by $W(\xi; a)$ the minimum value attained in Problem M if it exists, respectively.

We first show the following result.

Lemma B1. *Under Assumption B1,* $\Gamma(\xi; a) \neq \emptyset$ *for a large* $\xi > 0$.

Proof. Note that IC and LL constitute a system of linear inequalities with respect to $(b(z^1), ..., b(z^l))$, where $l \equiv \#Z$ denotes the number of all possible signals. Then, using the result for the existence of solutions to the system of linear inequalities with non-negativity constraints (see Avis and Kaluzny 2004), we can find a non-negative vector $(b(z^1), ..., b(z^l))$ that satisfies IC and LL under Assumption B1. Then, by adding a non-negative constant of $B \geq 0$ to each $b(z^k)$, we can also ensure that IR-M is satisfied without violating IC and LL. Let $\{b(z)\}_{z \in Z}$ be such a wage scheme. Then, if ξ is so large that $\xi \geq \max_{z \in Z} b(z)$, then DE-B holds as well. Q.E.D.

When ξ is so small, $\Gamma(\xi; a)$ is empty.[31] We define $\underline{\xi}(a) > 0$ as the lowest value of ξ such that

[31] For example, if $\xi < c(a)$, there are no wage schemes $(b(z))_{z \in Z}$ that satisfy IC, IR-M, LL, and DE-B.

$\Gamma(\xi;a)\neq\emptyset$. This $\underline{\xi}(a)>0$ is unique.[32] We define $W(\xi;a)=\infty$ when $\Gamma(\xi;a)=\emptyset$. $W(\xi;a)$ is also non-decreasing in ξ.

We next show the following.

Lemma B2. *$\Gamma(\xi;a)$ is a continuous correspondence at any $\xi\geq\underline{\xi}(a)$.*

Proof. We first show that $\Gamma(\xi;a)$ is upper hemicontinuous at $\xi\geq\underline{\xi}(a)$. Take any sequence $\{\xi^n\}$ such that $\xi^n\geq\underline{\xi}(a)$ for any n. Further, take any sequence $\{b^n\}$ such that $b^n\in\Gamma(\xi^n;a)$ for each n. Let $\xi^n\to\xi^\infty$ as $n\to\infty$ and $b^n\to b^\infty$. Then, suppose that $b^\infty\notin\Gamma(\xi^\infty;a)$. This implies that at least one of IC, IR-M, LL, and DE-B is not satisfied under ξ^∞. For example, suppose that IC is not satisfied:

$$\sum_z p(z|a)b^\infty(z)-c(a)<\sum_z p(z|a'')b^\infty(z)-c(a'')$$

for some $a''\neq a$. However, if we take a large enough n such that $b^n\simeq b^\infty$, by continuity, it must be that

$$\sum_z p(z|a)b^n(z)-c(a)<\sum_z p(z|a'')b^n(z)-c(a''),$$

which contradicts the fact that IC is satisfied at b^n. We can apply similar arguments for the cases that IR-M, LL, or DE-B do not hold. Thus, we must have $b^\infty\in\Gamma(\xi^\infty;a)$, which proves that $\Gamma(\cdot;a)$ is upper hemicontinuous at any $\xi\geq\underline{\xi}(a)$.

Next, we show that Γ is lower hemicontinuous at any $\xi\geq\underline{\xi}(a)$. Take any $\xi^0\geq\underline{\xi}(a)$ and any sequence $\{\xi^n\}$ such that $\xi^n\geq\underline{\xi}(a)$ for each n and $\xi^n\to\xi^0$ as $n\to\infty$. Choose any $b^0\in\Gamma(\xi^0;a)$. Then, we show the existence of a sequence $\{b^n\}$ such that $b^n\in\Gamma(\xi^n;a)$ for each n and $b^n\to b^0$ as $n\to\infty$.

Case 1: $\xi^0>\underline{\xi}(a)$. When $\xi^n>\underline{\xi}(a)$, we can choose $\tilde{b}^n\in\Gamma((1-\rho^n)\xi^n;a)$ for a small $\rho^n\in(0,1)$. Since $\xi^n>\underline{\xi}(a)$, such $\rho^n\in(0,1)$ exists, ensuring that $\Gamma((1-\rho^n)\xi^n;a)\neq\emptyset$. Define

$$\alpha^n\equiv\frac{\max\{\xi^0-\xi^n,0\}}{\xi^0-(1-\rho^n)\xi^n}$$

which belongs to $[0,1]$. Then, we define the following sequence of wage schemes $\{b^n\}$:

$$b^n(z)\equiv\begin{cases}\alpha^n\tilde{b}^n(z)+(1-\alpha^n)b^0(z) & \text{if } \xi^n>\underline{\xi}(a)\\ \text{some } \ \underline{b}\in\Gamma(\underline{\xi}(a);a) & \text{if } \xi^n=\underline{\xi}(a)\end{cases}$$

The wage scheme b^n therefore satisfies IC, IR-M, LL, and DE-B, as follows: first, if $\xi^n=\underline{\xi}(a)$,

[32]Suppose that $\Gamma(\xi;a)=\emptyset$ for some $\underline{\xi}(a)>0$. Then, for all $\xi<\underline{\xi}(a)$, we still have $\Gamma(\xi;a)=\emptyset$. In addition, if $\Gamma(\xi;a)\neq\emptyset$ for some ξ, then we still have $\Gamma(\xi'';a)\neq\emptyset$ for all $\xi''>\xi$.

then by definition, $\underline{b}$ satisfies IC, IR-M, LL, and DE-B. Second, if $\xi^n > \underline{\xi}(a)$, then we have

$$\begin{aligned}\sum_z p(z|a)b^n(z) - c(a) &= \alpha^n \left\{\sum_z p(z|a)\tilde{b}^n(z) - c(a)\right\} + (1-\alpha^n)\left\{\sum_z p(z|a)b^0(z) - c(a)\right\} \\ &\geq \alpha^n \left\{\sum_z p(z|a'')\tilde{b}^n(z) - c(a'')\right\} + (1-\alpha^n)\left\{\sum_z p(z|a'')b^0(z) - c(a'')\right\} \\ &= \sum_z p(z|a'')b^n(z) - c(a'')\end{aligned}$$

for any $a'' \neq a$. Thus, IC is satisfied by b^n. In addition, IR-M holds because

$$\begin{aligned}\sum_z p(z|a)b^n(z) - c(a) &= \alpha^n \left\{\sum_z p(z|a)\tilde{b}^n(z) - c(a)\right\} + (1-\alpha^n)\left\{\sum_z p(z|a)b^0(z) - c(a)\right\} \\ &\geq \alpha^n \times 0 + (1-\alpha^n) \times 0 \\ &= 0.\end{aligned}$$

We also obtain LL: $b^n(z) = \alpha^n \tilde{b}^n(z) + (1-\alpha^n)b^0(z) \geq 0$ for all $z \in Z$. Regarding DE-B, if $\xi^0 < \xi^n$, then $\alpha^n = 0$, so $b^n(z) = b^0(z) \leq \xi^0 < \xi^n$, and hence $b^n(z) \leq \xi^n$. Additionally, if $\xi^0 \geq \xi^n$, we obtain $\alpha^n \tilde{b}^n(z) + (1-\alpha^n)b^0(z) \leq \alpha^n(1-\rho^n)\xi^n + (1-\alpha^n)\xi^0 = \xi^n$ by the definition of α^n. Thus, $b^n(z) \leq \xi^n$. In either case, $b^n(z) \leq \xi^n$ for all $z \in Z$, thus proving DE-B. Finally, letting $n \to \infty$, we have $\xi^n \to \xi^0 > \underline{\xi}(a)$. Thus, $\xi^n > \underline{\xi}(a)$ for all large n, implying that $b^n(z) = \alpha^n \tilde{b}^n(z) + (1-\alpha^n)b^0(z)$ for all large n. Then, since $\alpha^n \to 0$ as $\xi^n \to \xi^0$, we have $b^n \to b^0$.

Case 2: $\xi^0 = \underline{\xi}(a)$. Take any sequence $\xi^n \to \xi^0 = \underline{\xi}(a)$, where $\xi^n \geq \underline{\xi}(a)$ for each n. Additionally, take any $b^0 \in \Gamma(\xi^0; a) = \Gamma(\underline{\xi}(a); a)$. Then, we choose $b^n = b^0 \in \Gamma(\underline{\xi}(a); a)$ for all n. This sequence $\{b^n\}$ trivially satisfies IC, IR-M, and LL. Furthermore, since $\xi^n \geq \underline{\xi}(a)$, we have $b^n(z) = b^0(z) \leq \underline{\xi}(a) \leq \xi^n$ such that $b^n(z) \leq \xi^n$ for all n. Since $b^n = b^0$ for all n and all $z \in Z$, this is the desired sequence.

The above argument shows that $\Gamma(\cdot; a)$ is lower hemicontinuous at any $\xi \geq \underline{\xi}(a)$. Q.E.D.

Since the objective function in Problem M is continuous with $b(z)$ and $\Gamma(\xi; a)$ is a compact and non-empty set for any given $\xi \geq \underline{\xi}(a)$,[33] Lemma B2 and Berge's Maximum Theorem implies that the optimal value $W(\xi; a)$ is a continuous function of ξ at any $\xi \geq \underline{\xi}(a)$.

We also show the following.

Lemma B3. *$W(\xi; a)$ is a convex function of ξ.*

[33] Since $\xi \geq b(z) \geq 0$ holds, $\Gamma(\xi; a)$ is bounded. Additionally, we show that $\Gamma(\xi; a)$ is closed using the continuity of all constraints with respect to $b(z)$ and ξ.

Proof. Let ξ'' and ξ', and b'' and b' be the optimal contracts that solve Problem M that implements $a \in \mathbb{A}$ given ξ'' and ξ', respectively. Then, let $\tilde{\xi} \equiv \beta\xi'' + (1-\beta)\xi'$ for any $\beta \in (0,1)$, and consider the wage scheme $\tilde{b} \equiv \beta b'' + (1-\beta)b'$ given $\tilde{\xi}$ in Problem M that implements $a \in \mathbb{A}$. By the definitions of b'' and b', they satisfy IC, IR-M, and LL. Then, since managers are risk neutral, $\tilde{b}$ satisfies IC and IR-M as well. $\tilde{b}$ also satisfies LL. In addition, $\tilde{b}(z) = \beta b''(z) + (1-\beta)b'(z) \leq \beta\xi'' + (1-\beta)\xi' = \tilde{\xi}$ such that $\tilde{b}(z) \leq \tilde{\xi}$, implying that $\tilde{b}$ satisfies DE given $\tilde{\xi}$. Thus, $\tilde{b}$ is feasible in Problem M with $\tilde{\xi}$. This implies that

$$\begin{aligned} W(\tilde{\xi}; a) &\leq \sum_z p(z|a)\tilde{b}(z) \\ &= \sum_z p(z|a)\{\beta b''(z) + (1-\beta)b'(z)\} \\ &= \beta \sum_z p(z|a)b''(z) + (1-\beta)\sum_z p(z|a)b'(z) \\ &= \beta W(\xi''; a) + (1-\beta)W(\xi'; a) \end{aligned}$$

which shows that W is a convex function of ξ. Q.E.D.

When implementing the management quality level a_t in period t, we must have $V_t - k_t - L \geq \underline{\xi}(a_t)$. Thus, by setting $\xi = V_t - L - k_t$ in $W(\xi; a_t)$, the minimum expected wage for implementing a_t from the old manager is given by $W(V_t - L - k_t; a_t)$. Then, the old entrepreneur chooses the widget production level k_t to maximize the following profit, defined as the output y_t minus the payment to the widget producer k_t and the old manager's expected wage $W(V_t - L - k_t; a_t)$:

$$F(h(a_t, A_{t-1}), k_t) - k_t - W(V_t - L - k_t; a_t) \tag{B1}$$

subject to $V_t - L - \underline{\xi}(a) \geq k_t \geq 0$, given A_{t-1}, V_t, and a_t. Since $W(V_t - L - k_t; a)$ is continuous and convex with $V_t - L - k_t$, W is continuous and convex with k_t as well. Then, there exists a unique optimal of k_t that maximizes the above profit (B1) subject to $V_t - L - \underline{\xi}(a) \geq k_t \geq 0$. We define the old entrepreneur's maximum profit from having the old manager implement management quality level a_t in period t as

$$\tilde{\Phi}(a_t; A_{t-1}, V_t) \equiv \max\left\{ \max_{k_t : V_t - L - \underline{\xi}(a_t) \geq k_t \geq 0} F(h(a_t, A_{t-1}), k_t) - k_t - W(V_t - L - k_t; a_t), 0 \right\} \tag{B2}$$

Note that when $V_t = L + \underline{\xi}(a_t)$, we must have $k_t = 0$ because $V_t - L - \underline{\xi}(a_t) \geq k_t \geq 0$. Thus, at $V_t = L + \underline{\xi}(a_t)$, the firm's profit must be non-positive:

$$F(h(a_t, A_{t-1}), 0) - W(\underline{\xi}(a_t); a_t) = -W(\underline{\xi}(a_t); a_t) \leq 0$$

where $W(\underline{\xi}(a); a) \geq c(a) \geq 0$. Then, for all V_t greater than but close to $L + \underline{\xi}(a_t)$, the profit above becomes zero; that is, $\tilde{\Phi}(a^i; V_t, A_{t-1}) = 0$ (Figure B1). Moreover, since F and W are continuous in k_t and V_t,[34] by the Berge's Maximum Theorem, $\tilde{\Phi}$ is continuous in $V_t \in [L, I]$.

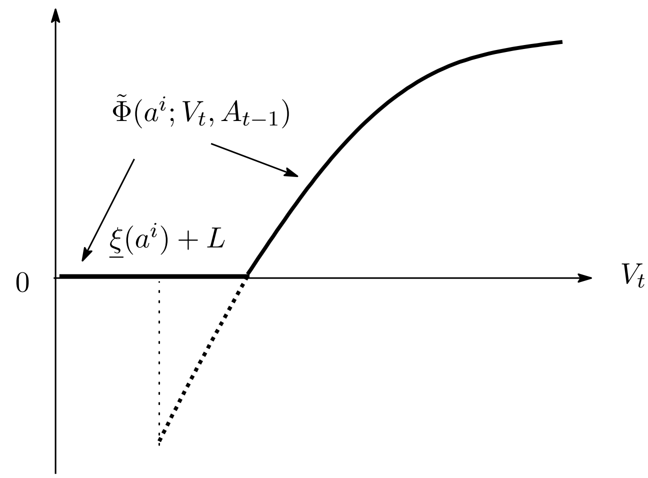

Figure B1: firm's profit as a function of V_t

Given the above result, each old entrepreneur chooses a management quality of a_t to maximize her profit $\tilde{\Phi}(a_t; A_{t-1}, V_t)$ over all possible management quality levels $\mathbb{A} \equiv \{a^0, ..., a^m\}$. With a slight abuse of notation and the use of the same notation as in the basic model, we define the overall maximum profit attained by the old entrepreneur in period t as

$$\Phi(V_t, A_{t-1}) \equiv \max_{a \in \mathbb{A}} \tilde{\Phi}(a; A_{t-1}, V_t). \tag{B3}$$

Since $\tilde{\Phi}$ is continuous in V_t and the set of possible management quality levels $\mathbb{A}$ is finite, Φ is continuous in V_t.

To extend Propositions 1 and 2, we replace $c(a_t)$ by $W(V_t - L - k_t; a_t)$ in the proofs of Propositions 1 and 2. We also replace the upper bound for k_t given in the proofs of Propositions 1 and 2, $V_t - L - c(a_t)$, by $V_t - L - \underline{\xi}(a_t)$. Then, since we already established the continuity of $W(\cdot; a)$ above, the profit function Φ is continuous as well. We can hence follow the same steps as in the proofs of Propositions 1 and 2 and obtain results similar to those of Propositions 1 and 2.

Proposition B1. *Suppose that $\delta \to 1$ and $a^0 \to 0$. Then, every equilibrium path becomes cyclical.*

[34]Evidently, the constraint set $[0, V_t - L - \underline{\xi}(a_t)]$ for the choice variable k_t is a compact and continuous correspondence.

Proposition B2. *Suppose that $A_0 \in (A^0, A^m)$, $\delta \to 1$, and $a^0 \to 0$. Then, there exists a cyclical equilibrium.*

11 Appendix C: Contingent Payment to Widget Producers

In Appendix B, we assumed that the payment to a widget producer R_t does not depend on the performance signal $z_t \in Z$ of the manager's choice of management quality a_t. In this appendix, we extend the model to allow the payment to the widget producer R_t to depend on the manager's performance signal z_t. $R_t(z)$ denotes the payment to the widget producer, contingent on the signal $z \in Z$. The main role of these contingent payments to widget producers is to relax the DE constraint.

When R_t varies with the performance signal z, we modify the DE constraint:

$$V_t - L \geq R(z_t) + b_t(z), \quad \text{for all } z \in Z. \qquad \text{(DE}^*\text{)}$$

Since the original DE-B constraint given in Appendix B requires a constant payment to the widget producer, R_t, at $R_t = k_t$ for all $z \in Z$, the modified constraint above is weaker than the original one.

One possible technical difficulty for this extension is that the firm's profit function $\tilde{\Phi}(a_t, V_t, A_{t-1})$ is not necessarily continuous in its market value V_t as we will see below. This may cause the technical problem of showing the existence of the equilibrium paths illustrated in Proposition B2. To recover the continuity, we slightly modify the basic model by introducing a fixed cost of widget production, $f > 0$. Each widget producer can produce a positive level of widget $k_t > 0$ at a cost of $k_t + f$. We will discuss why and how such a fixed cost is related to the continuity of the firm's profit function later. The rough intuition for this result is that, if no fixed costs are present, a positive production of widget $k_t > 0$ is still possible even at the lowest firm value $V_t = L + \underline{\xi}(a_t)$ to implement a_t, in contrast to the case of non-contingent payments.[35] Then, the firm's profit may jump down at the lowest firm value $V_t = L + \underline{\xi}(a_t)$ because for any $V < L + \underline{\xi}(a_t)$, it is impossible to implement a_t, so $\tilde{\Phi}(a_t; V, A_{t-1}) = 0$ holds by definition.

In the following, we first show that we can extend Proposition B1 by allowing contingent payments to widget producers. Second, we show the existence of cyclical equilibrium paths as a counterpart to Proposition B2. Although we do not need the fixed cost of $f > 0$ to show the former result, we introduce the fixed cost at the outset of this extension to avoid repetitive explanations.

[35]When $V_t = L + \underline{\xi}(a_t)$, $R_t = 0$ must hold so that $k_t = 0$ holds in the model in Appendix B.

We consider the feasible set of $k_t \geq 0$, R_t, and b_t, denoted by $\Omega(\xi, a_t)$, which satisfies all of the following constraints:

$$\sum_z p(z|a_t)b_t(z) - c(a_t) \geq \sum_z p(z|a'')b_t(z) - c(a''), \quad \text{for any } a'' \neq a_t \tag{IC}$$

$$\sum_z p(z|a_t)b_t(z) - c(a_t) \geq 0 \tag{IR-M}$$

$$\xi \geq b_t(z) + R_t(z) \;\; \forall\, z \in Z \tag{DE*}$$

$$\sum_z p(z|a_t)R_t(z) \geq k_t + f \tag{IR-W}$$

$$b_t(z) \geq 0, \quad R_t(z) \geq 0 \;\; \forall\, z \in Z \tag{LL}$$

Here, we add $f > 0$ even when $k_t = 0$ only for convenience.[36]

We next show Lemma C1.

Lemma C1. *$\Omega(\xi; a) \neq \emptyset$ for a large ξ.*

Proof. Let DE* be dropped. Then, under Assumption B1, we can find some wage scheme b_t for the manager such that IC-M, IR-M, and LL in which all $b_t(z) \geq 0$ are satisfied.[37] Then, we can set R_t to satisfy $\sum_z p(z|a_t)R_t(z) = k_t + f$ and LL where $R_t(z) \geq 0$. When ξ is large enough, these payments satisfy DE* as well. Q.E.D.

Lemma C2. *$\Omega(\xi; a)$ is a compact set.*

Proof. Note that $b_t(z) \geq 0$, $R_t(z) \geq 0$ and $k_t \geq 0$. In addition, it follows from DE* that $b_t(z) \leq \xi$ and $R_t(z) \leq \xi$. We also have $k_t \leq \sum_z p(z|a_t)R_t(z) - f \leq \xi - f$ and hence $k_t \leq \xi - f$. Thus, $\Omega(\xi; a)$ is bounded. By the standard argument of continuity, $\Omega(\xi; a)$ is closed. Q.E.D.

We next define the lowest value of ξ such that $\Omega(\xi, a_t) \neq \emptyset$ for a given a_t. When ξ is so small that $\xi < c(a_t) + f$, $\Omega(\xi, a_t) = \emptyset$. Thus, there exists some $\hat{\xi}(a)$ such that $\Omega(\xi, a_t) \neq \emptyset$ for all $\xi \geq \hat{\xi}(a_t)$. Since any wage scheme b_t belonging to $\Omega(\xi, a_t)$ satisfies all IC, IR-M, LL, and $\xi \geq b_t(z)$, this b_t belongs to $\Gamma(\xi; a_t)$ as well (recall that $\Gamma(\xi; a_t)$ is the set of feasible contracts satisfying IC-M, IR-M, IR-W and DE, given in the main text.) Thus $\hat{\xi}(a_t) \geq \underline{\xi}(a_t)$.

[36] We use this formulation only to prove Lemma C5 and Lemma C6 below which we utilize for the proof of Proposition C2. These lemmas show that $k_t = 0$ holds if and only if ξ is larger than or equal to some cut off value, which we will define as $\hat{\xi}(a)$ below, such that $\Omega(\xi, a) \neq \emptyset$ if and only if $\xi \geq \hat{\xi}(a)$. However, since firm's profits become negative when k is positive but small, we can rule out the case that $k_t = 0$ happens in equilibrium paths, meaning that $V_t - L$ is always greater than the cut off value $\hat{\xi}(a_t)$.

[37] See Avis and Kaluzny (2004) for the mathematical result on the existence of solutions to a system of linear inequalities with non-negative constraints.

Lemma C3. $\Omega(\xi; a)$ *is a continuous correspondence at any* $\xi \geq \hat{\xi}(a)$.

Proof. $\hat{\xi}(a)$ is the lowest value of ξ for which $\Omega(\xi; a) \neq \emptyset$. That is, $\Omega(\xi; a) \neq \emptyset$ holds if and only if $\xi \geq \hat{\xi}(a)$.

(i) Upper hemicontinuity: Take any $\{b^n, R^n, k^n\} \in \Omega(\xi^n; a)$ and let $\xi^n \to \xi^\infty \geq \hat{\xi}(a)$. Let also $(b^n, R^n, k^n) \to (b^\infty, R^\infty, k^\infty)$. Then, by the continuity argument, we have $(b^\infty, R^\infty, k^\infty) \in \Omega(\xi^\infty; a)$.

(ii) Lower hemicontinuity: Take any $(b^0, R^0, k^0) \in \Omega(\xi^0, a)$ and any sequence $\{\xi^n\}$ such that $\xi^n \to \xi^0$.

Case 1: $\xi^0 > \hat{\xi}(a)$. When $\xi^n > \hat{\xi}(a)$, we can choose some $\rho^n \in (0,1)$ such that $\Omega((1-\rho^n)\xi^n; a) \neq \emptyset$. In particular, since $\xi^0 > \hat{\xi}(a)$ as we assumed, we can find some $\rho^0 \in (0,1)$ such that $\Omega((1-\rho^0)\xi^0; a) \neq \emptyset$.

Then, we define

$$\alpha^n \equiv \frac{\max\{\xi^0 - \xi^n, 0\}}{\xi^0 - (1-\rho^n)\xi^n}$$

which belongs to $[0,1]$. When $\xi^n > \hat{\xi}(a)$, we can take some $(\tilde{b}^n, \tilde{R}^n, \tilde{k}^n) \in \Omega((1-\rho^n)\xi^n; a)$. We also choose $(\underline{b}, \underline{R}, \underline{k}) \in \Omega(\hat{\xi}(a); a)$.

We define the following sequence:

$$b^n(z) \equiv \begin{cases} \alpha^n \tilde{b}^n(z) + (1-\alpha^n) b^0(z) & \text{if } \xi^n > \hat{\xi}(a) \\ \underline{b}(z) & \text{if } \xi^n = \hat{\xi}(a) \end{cases}$$

$$R^n(z) \equiv \begin{cases} \alpha^n \tilde{R}^n(z) + (1-\alpha^n) R^0(z) & \text{if } \xi^n > \hat{\xi}(a) \\ \underline{R}(z) & \text{if } \xi^n = \hat{\xi}(a) \end{cases}$$

$$k^n \equiv \begin{cases} \alpha^n \tilde{k}^n + (1-\alpha^n) k^0 & \text{if } \xi^n > \hat{\xi}(a) \\ \underline{k} & \text{if } \xi^n = \hat{\xi}(a) \end{cases}$$

First, we show that $(b^n, R^n, k^n) \in \Omega(\xi^n; a)$ for each n. Regarding IC-M, we can verify that when $\xi^n > \hat{\xi}(a)$,

$$\begin{aligned} \sum_z p(z|a) b^n(z) &= \alpha^n \{\sum_z p(z|a) - p(z|a'')\} \tilde{b}^n(z) + (1-\alpha^n) \sum_z \{p(z|a) - p(z|a'')\} b^0(z) \\ &\geq \alpha^n \{c(a) - c(a'')\} + (1-\alpha^n)\{c(a) - c(a'')\} \\ &= c(a) - c(a'') \end{aligned}$$

for all $a'' \neq a$. When $\xi^n = \hat{\xi}(a)$, we have

$$\sum_z \{p(z|a) - p(z|a'')\} b^n(z) = \sum_z \{p(z|a) - p(z|a'')\} \underline{b}(z) \geq c(a) - c(a'')$$

for all $a'' \neq a$, which proves IC-M. Second, we can similarly show that IR-M is satisfied. Third, we have

$$\begin{aligned}\sum_z R^n(z) &= \alpha^n \sum_z p(z|a)\tilde{R}^n(z) + (1-\alpha^n)\sum_z p(z|a)R^0(z) \\ &\geq \alpha^n \tilde{k}^n + (1-\alpha^n)k^0 + f \\ &= k^n + f\end{aligned}$$

when $\xi^n > \hat{\xi}(a)$. When $\xi^n = \hat{\xi}(a)$, we obtain

$$\begin{aligned}\sum_z p(z|a)R^n(z) &= \sum_z p(z|a)\underline{R}(z) \\ &\geq \underline{k} + f \\ &= k^n + f\end{aligned}$$

Thus, we obtain IR-W. Fourth, regarding DE*, we can show the following: when $\xi^n > \xi^0$, $\alpha^n = 0$ holds so that $(b^n, R^n, k^n) = (b^0, R^0, k^0)$. Then, $b^n(z) + R^n(z) = b^0(z) + R^0(z) \leq \xi^0 < \xi^n$. Thus, $b^n(z) + R^n(z) \leq \xi^n$. When $\xi^n < \xi^0$,

$$\alpha^n = \frac{\xi^0 - \xi^n}{\xi^0 - (1-\rho^n)\xi^n} > 0$$

so that, by the definition of α^n, we obtain

$$\begin{aligned}b^n(z) + R^n(z) &= \alpha^n(\tilde{b}^n(z) + \tilde{R}^n(z)) + (1-\alpha^n)(b^0(z) + R^0(z)) \\ &\leq \alpha^n(1-\rho^n)\xi^n + (1-\alpha^n)\xi^0 \\ &= \xi^n\end{aligned}$$

for all $z \in Z$.

The sequence above converges to (b^0, R^0, k^0) as $n \to \infty$ because $\alpha^0 \to 0$ as $n \to \infty$.

Case 2: $\xi^0 = \hat{\xi}(a)$. Take any $\xi^n \to V^0 = \hat{\xi}(a)$. Then, choose any $(b^0, R^0, k^0) \in \Omega(\hat{\xi}(a); a)$. Let $(b^n, R^n, k^n) = (b^0, R^0, k^0)$ for each n. This sequence trivially satisfies all IC-M, IR-M, IR-W, LL, and DE*. In addition, since $\xi^n \geq \hat{\xi}(a)$, we have $b^n(z) + R^n(z) = b^0(z) + R^0(z) \leq \hat{\xi}(a) \leq V^n$ for all $z \in Z$, which shows that DE* is satisfied. This sequence trivially converges to (b^0, R^0, k^0). Q.E.D.

By setting $\xi = V_t - L$, we consider the optimization problem that the old entrepreneur should solve in period t given a_t, A_{t-1}, and V_t:

Problem P.

$$\max_{k_t \geq 0, b_t, R_t} \; F(A_t, k_t) - \sum_z p(z|a_t)\{b_t(z) + R_t(z)\}$$

subject to $\{k_t, R_t, b_t\} \in \Omega(V_t - L, a_t)$.

We then define the optimal value of Problem P as

$$\tilde{\Phi}(a_t; V_t, A_{t-1}) \equiv \max\left\{ \max_{\{k_t,R_t,b_t\}\in\Omega(V_t-L;a_t)} F(h(a_t, A_{t-1}), k_t) - \sum_z p(z|a_t)\{b_t(z) + R_t(z)\}, 0 \right\}.$$

Since the constraint set $\Omega(V_t - L; a)$ is non-empty for $V \geq L + \hat{\xi}(a)$ and compact, the above optimal value $\tilde{\Phi}$ exists. We define $\tilde{\Phi}(a; V, A) = 0$ if $\Omega(V - L; a) = \emptyset$. Further, since the constraint set $\Omega(V - L; a)$ is a continuous correspondence at any $V \geq L + \hat{\xi}(a)$ and the objective function is continuous, by the Berge's Maximum Theorem, the optimal value $\tilde{\Phi}$ is continuous in $V \geq L + \hat{\xi}(a)$. We then define the maximum profit that the old entrepreneur can attain in period t as

$$\Phi(V_t, A_{t-1}) \equiv \max_{a\in\mathbb{A}} \tilde{\Phi}(a; V_t, A_{t-1}).$$

The difference from the case of non-contingent payment in Appendix B is that $R(z)$ is not necessarily zero for all $z \in Z$, even when $V = L + \hat{\xi}(a)$. To see this, suppose a sufficiently small fixed cost of f. Specifically, we set $f = 0$. At $V = L + \hat{\xi}(a)$ for $a \neq a^0$, there exists some $z \in Z$ such that $b(z) < \hat{\xi}(a)$[38]. Then we can choose some positive $k > 0$ and the corresponding payment R such that $\sum_z p(z|a)R(z) = k$ and DE-B hold together: $\hat{\xi}(a) \geq b(z) + R(z)$ for all $z \in Z$ when $k \to 0$. Thus, the entrepreneur can implement a positive production level of widgets, $k > 0$, even at $V = L + \hat{\xi}(a)$, which implies that $\tilde{\Phi}(a_t; L + \hat{\xi}(a_t), A_{t-1}) > 0$ may happen. This may cause the discontinuity of $\Phi(V_t, A_{t-1})$ at $V = L + \hat{\xi}(a_t)$ because $\Omega(V_t - L; a_t) = \emptyset$ holds when $V < L + \hat{\xi}(a_t)$ (see Figure C1). This argument still holds when there is a positive but small fixed cost of $f > 0$.

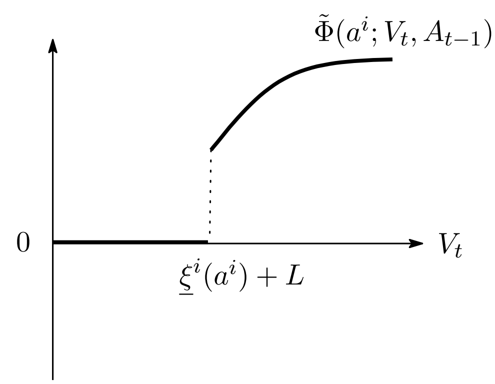

Figure C1: discontinuity of firm's profit

[38]For $a \neq a^0$, $\Omega(\hat{\xi}(a); a) \neq \emptyset$ implies that IC is satisfied at $V = L + \hat{\xi}(a)$, so $b(z) < \hat{\xi}(a)$ must be satisfied for some $z \in Z$. This is because if $b(z) = \hat{\xi}(a)$ for all $z \in Z$, then IC-M is never satisfied for $a \neq a^0$. Thus, $b(z) < \hat{\xi}(a)$ must hold for some $z \in Z$.

Despite such discontinuity, we can show the counterpart to Proposition B1.

Proposition C1. *Consider the case in which payments to widget producers $R(z)$ can depend on the performance signals $z \in Z$ of the manager's choice of management quality. Suppose that $\delta \in (0,1)$ is sufficiently close to 1 and that a^0 is close to zero. Then, every equilibrium path becomes cyclical.*

Proof. Suppose, contrary to the claim, that there exists an equilibrium that is convergent in any good management quality $a^i \neq a^0$. That is, $A_t \to A^i$ as $t \to \infty$. Thus, $|A_t - A^i| < \varepsilon$ for large enough t given a small $\varepsilon > 0$. In such an equilibrium, the old entrepreneur's profit becomes

$$\Phi(V_t, A_{t-1}) = \tilde{\Phi}(a^i; V_t, A_{t-1}) \in (\tilde{\Phi}(a^i; V_t, A^i - \varepsilon), \tilde{\Phi}(a^i; V_t, A^i + \varepsilon))$$

for all large enough t for a given $\varepsilon > 0$. At $V_t = L + \underline{\xi}(a^i)$, $\tilde{\Phi}(a^i; V_t, A_{t-1})$ may be discontinuous. In particular, when $\delta \to 1$ such that $A^i \to \infty$, $\tilde{\Phi}(a^i; L + \hat{\xi}(a^i), A^i)$ may take a large value such that $\tilde{\Phi}(a^i; L+\hat{\xi}(a^i), A^i) + \hat{\xi}(a^i) + L > r(L+\hat{\xi}(a^i))$. Thus, when $\delta \to 1$, $\tilde{\Phi}(a^i; V, A)$ may jump up at $V = L + \hat{\xi}(a^i)$. When there are no such discontinuity, we can apply the same argument as in the proof in Proposition B1. Thus, we consider only the case in which a discontinuity at $V = L + \hat{\xi}(a^i)$ exists. In this case, the firm's market value evolves according to

$$rV_{t-1} = \begin{cases} \tilde{\Phi}(a^i; V_t, A_{t-1}) + V_t & \text{if } V_t \geq L + \hat{\xi}(a^i) \\ V_t & \text{otherwise} \end{cases}$$

In addition, when $\delta \to 1$, any value V satisfying $rV = \tilde{\Phi}(a^i; V, A^i - \varepsilon) + V$ must satisfy $V > I$. Thus, any path $\{V_t\}$ that starts from $V_0 \in [L, I]$ must eventually have $V_t < L + \hat{\xi}(a^i)$ (Figure C2), which, however, cannot be an equilibrium.

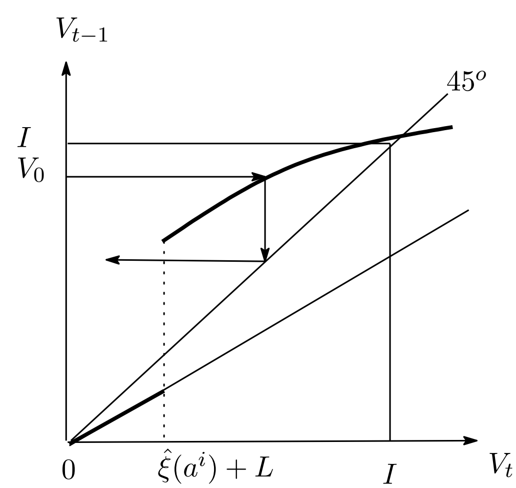

Figure C2: no equilibrium paths which are convergent in $a^i \neq a^0$.

Second, suppose the existence of an equilibrium path that is convergent in the lowest management quality a^0. Thus, $A_t \to A^0$ as $t \to \infty$ such that $|A_t - A^0| < \varepsilon$ for a large t given a small $\varepsilon > 0$. In such an equilibrium, the entrepreneur's flow profit becomes $\tilde{\Phi}(a^0; V_t, A_{t-1})$, which is bounded above by $\tilde{\Phi}(a^0; V_t, A^0 + \varepsilon)$ for large enough t for a given small $\varepsilon > 0$. When $a^0 \to 0$ and hence $A^0 \to 0$, this profit tends to be zero by taking $\varepsilon \to 0$; hence, the firm's market value evolves according to $rV_{t-1} = V_t$, implying that $V_t \to \infty$ as $t \to \infty$. This also cannot be an equilibrium. Q.E.D.

We next show the counterpart to Proposition B2 that there exists a cyclical equilibrium path. As we mentioned, one technical difficulty for this purpose is that the firm's profit function $\tilde{\Phi}(a, \cdot, A)$ is not necessarily continuous in the firm value V at $V = L + \hat{\xi}(a)$. We rule out the possibility of such discontinuity by imposing the following assumption.

Assumption C1. *$f > \underline{\xi}(a) - c(a)$ for all $a \in \mathbb{A}$.*

Under Assumption C1, we can ensure that $\hat{\xi}(a) > \underline{\xi}(a)$ for any $a \in \mathbb{A}$.[39]

We show the following claim.

Lemma C5. *Take any $\{k, R, b\} \in \Omega(\xi; a)$. Then, $k = 0$ must hold at $\xi = \hat{\xi}(a)$.*

Proof. First, note that $\Gamma(\xi; a)$ is a continuous correspondence at any $\xi \geq \underline{\xi}(a)$ (see Lemma B2). Specifically, it is a lower hemicontinuous. Since $\hat{\xi}(a) > \underline{\xi}(a)$ under Assumption C1, $\Gamma(\hat{\xi}(a); a) \neq \emptyset$. Then, for any $\hat{b} \in \Gamma(\hat{\xi}(a); a)$ and any sequence $\{\xi^n\}$ such that $\underline{\xi}(a) < \xi^n < \hat{\xi}(a)$ for each n with $\xi^n \to \hat{\xi}(a)$, there exists a sequence $\{b^n\}$ such that $b^n \in \Gamma(\xi^n; a)$ and $b^n \to \hat{b}$.

Now, take any $\{\hat{b}, \hat{R}, \hat{k}\} \in \Omega(\hat{\xi}(a); a)$ and suppose the opposite of the claim that $\hat{k} > 0$. By definition, $\hat{\xi}(a) \geq \hat{b}(z) + \hat{R}(z)$ for all $z \in Z$. Then, we choose $\tilde{k} > 0$ which is slightly lower than $\hat{k} > 0$. We also correspondingly set $\tilde{R}(z) \equiv \hat{R}(z) - \varepsilon$ for a small given $\varepsilon > 0$ for any $z \in Z$ such that $\hat{R}(z) > 0$, while also satisfying IR-W. We also take $b^n \in \Gamma(\xi^n; a)$ for the sequence $\{\xi^n\}$ such that $\underline{\xi}(a) < \xi^n < \hat{\xi}(a)$ for each n. Note that $b^n \to \hat{b}$ as $\xi^n \to \hat{\xi}(a)$, and that b^n satisfies all IC, IR-M, and LL.

Then, for small $\eta > 0$ and $\rho > 0$, we can take a large enough n to ensure that $\xi^n + \eta \geq \hat{\xi}(a)$

[39] DE*, IR-M, and IR-W imply that $\hat{\xi}(a) \geq \sum_z p(z|a)(b(z) + R(z)) \geq c(a) + k + f \geq c(a) + f$. Then, Assumption C1 shows that $\hat{\xi}(a) > \underline{\xi}(a)$.

and $\hat{b}(z) \geq b^n(z) - \rho$ for each z. This implies that

$$\begin{aligned} \xi^n + \eta &\geq \hat{\xi}(a) \\ &\geq \hat{b}(z) + \hat{R}(z) \\ &\geq b^n(z) - \rho + \tilde{R}(z) + \varepsilon, \end{aligned}$$

which shows that $\xi^n \geq b^n(z) + \tilde{R}(z) + \varepsilon - \eta - \rho$. We then take small $\eta \to 0$ and $\rho \to 0$ such that $\varepsilon \geq \rho + \eta$. Then, $\xi^n \geq b^n(z) + \tilde{R}(z)$ for all $z \in Z$. Take $\{b^n, \tilde{R}, \tilde{k}\}$ at ξ^n. Then, DE* is satisfied at ξ^n because $\xi^n \geq b^n(z) + \tilde{R}(z)$ for all $z \in Z$. Also, since all other constraints, IC, IR-M, LL, and IR-W are satisfied at ξ^n, we have $\Omega(\xi^n; a) \neq \emptyset$ and $\xi^n < \hat{\xi}(a)$. This contradicts the definition of $\hat{\xi}(a)$: the lowest value of ξ for which $\Omega(\xi; a) \neq \emptyset$. Q.E.D.

Lemma C6. *There exists some $\{k, R, b\} \in \Omega(\xi; a)$ such that $k > 0$ holds when $\xi > \hat{\xi}(a)$.*

Proof. Suppose that $\xi > \hat{\xi}(a)$. Take any $\{\hat{k}, \hat{R}, \hat{b}\} \in \Omega(\hat{\xi}(a); a)$. By Lemma C5, $\hat{k} = 0$ holds. Thus $\sum_z p(z|a)\hat{R}(z) \geq f$ holds. We set $R''(z) \equiv \hat{R}(z) + \rho$ for a small $\rho > 0$ for all $z \in Z$. Specifically, we take ρ such that $\xi - \hat{\xi}(a) \geq \rho$. Then, by considering $\{\hat{b}, R'', k''\}$, where $k'' > 0$ and $\sum_z p(z|a)R''(z) = k'' + f$, we can ensure that $\hat{b}(z) + R''(z) = \hat{b}(z) + \hat{R}(z) + \rho \leq \hat{\xi}(a) + \rho \leq \xi$, so $\xi \geq \hat{b}(z) + R''(z)$. Thus, DE* is satisfied. Note also that $\hat{b}$ satisfies IC, IR-M, and LL. Then, $\{\hat{b}, R'', k''\} \in \Omega(\xi; a)$ for $\xi > \hat{\xi}(a)$. Q.E.D.

By Lemma C5, we can show that for any $\{k, R, b\} \in \Omega(\hat{\xi}(a); a)$, we have

$$F(h(a, A), k) - \sum_z p(z|a)(b(z) + R(z)) \leq -\sum_z p(z|a)b(z) - f < 0 \tag{C1}$$

at $V = L + \hat{\xi}(a)$ because $k = 0$ holds at $V = L + \hat{\xi}(a)$ and hence we have $F(h(a, A), 0) = 0$. Also, by Lemma C6, $k > 0$ is possible for any $V > L + \hat{\xi}(a)$. We already know that the firm's profit function $\tilde{\Phi}(a; V, A)$ is continuous in $V \geq L + \hat{\xi}(a)$ for each $a \in \mathbb{A}$. By combining this with (C1) above, $\tilde{\Phi}(a; V, A)$ is continuous and non-decreasing in $V \geq L + \hat{\xi}(a)$, and $\tilde{\Phi}(a; V, A) \geq 0$ holds if and only if $V \geq \hat{V}(a, A)$ for some $\hat{V}(a, A) > L + \hat{\xi}(a)$ (see Figure C3). Thus, we can preserve the continuity property of $\tilde{\Phi}(a; \cdot, A)$ with respect to V, as in the case in Appendix B.

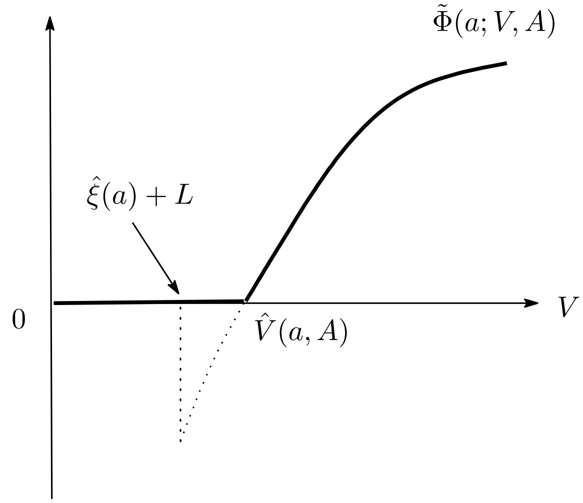

Figure C3: continuity of firm's profit

We make a similar assumption to Assumption 3 in the main text.

Assumption C2. *There exists some $a \neq a^0$ such that*

$$\max_{V \geq L+\hat{\xi}(a)} \tilde{\Phi}(a; V, A^0) - (r-1)V > 0.$$

This assumption holds when $h(a, A^0)$ is large enough for some $a \neq a^0$.

We define the overall maximum profit $\Phi(V, A) \equiv \max_{a \in \mathbb{A}} \tilde{\Phi}(a; V, A)$. Then, since $\tilde{\Phi}(a; \cdot, A)$ is continuous in V for each $a \in \mathbb{A}$, and $\mathbb{A}$ is a finite set, $\Phi(\cdot, A)$. In addition, under Assumption C2 above, $\Phi(V, A^0) + V > rV$ for some $V \geq L$. Moreover, $\Phi(V, A) + V < rV$ for all small $V \geq L$. Specifically, $\Phi(V, A) = 0$ holds for any $V < \min_{a \in \mathbb{A}} \hat{V}(a, A)$, where note that $\hat{V}(a, A) > L + \hat{\xi}(a) > L$ for all $a \in \mathbb{A}$. Thus, Φ is continuous in V.