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Michi NISHIHARA

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Graduate School of Economics
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

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Michi NISHIHARA[†]

Abstract

This paper presents a framework to study real options with illiquidity of option exercise opportunities. I incorporate a constraint that the investment time is chosen from Poisson arrival times in the standard real option value (ROV) model. I derive the closed-form solution and show that illiquidity decreases the option value of waiting and the investment threshold. I extend the results to a case with different types of projects and show that an inferior project can be undertaken in the presence of illiquidity. I prove that the solution of the illiquid model converges to that of the ROV model for higher liquidity and converges to that of the net present value (NPV) model for lower liquidity. I also show that the solution agrees with the limit of the corresponding regime-switching model. The results fill the gaps in the NPV, ROV, and regime-switching models and reveal the effects of illiquidity on investment decisions.

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[†]Corresponding Author. Graduate School of Economics, Osaka University, 1-7 Machikaneyama, Toyonaka, Osaka 560-0043, Japan, E-mail: nishihara@econ.osaka-u.ac.jp, Phone: 81-6-6850-5242, Fax: 81-6-6850-5277

1 Introduction

In project valuation involving high uncertainty and managerial flexibility, the real option value (ROV) method adds value to the basic net present value (NPV) method (cf. Dixit and Pindyck (1994)). A number of papers have recently investigated corporate financial issues, such as financing and investment, mergers and acquisitions, and bankruptcy and liquidation in the ROV framework. Most of these studies presume that an option holder can exercise his/her real option at an arbitrary time, analogous to American call and put options.

In the real world, however, the assumption of arbitrary exercise timing does not always hold. For example, consider a firm that will expand its business by acquiring (a certain section of) another firm. If a hostile acquisition is too costly, the firm begins by searching for a target that may potentially agree to be acquired. Only if the firm finds such a target and the negotiation and/or bidding process succeeds, the firm can acquire the target assets. Otherwise, the firm begins by searching for another target. In this case, the firm can exercise the acquisition option not at an arbitrary time but only when it finds a target satisfying certain conditions. Similarly, in an asset liquidation problem, a firm can sell illiquid assets at fair prices not at an arbitrary time but only when it meets a counterparty. Another example is a financially constrained firm's investment. A firm can invest in a project not at an arbitrary time but when it is able to raise funds for the project. Some investments might be feasible only when the investment programs qualify for government grants and subsidies. Boyle and Guthrie (2003) investigate investment timing decisions, assuming that investment is feasible only when internal funds are sufficient.

Real options frequently concern transactions of illiquid assets, such as real estate, technology, patents, and businesses. Capital market friction may also restrict real option exercise opportunities. Nevertheless, in the real options context, few papers have investigated the effects of illiquidity of exercise opportunities. On the other hand, a number of papers focus on market illiquidity of labor, real estate, and financial assets such as debts and derivatives, etc. For example, Duffie, Garleanu, and Pedersen (2005) reveal how intermediation and asset prices in over-the-counter markets are affected by illiquidity. He and Xiong (2012) and He and Milbradt (2014) show the effects of debt market illiquidity on a firm's bankruptcy timing. These studies are based on search theory, which was first proposed by Diamond (1982) to investigate labor market illiquidity.

This paper develops a new framework to study real option problems with illiquidity of option exercise opportunities. To do so, I incorporate a constraint that the investment time is chosen from Poisson arrival times in the investment timing model (hereafter, the ROV model) of McDonald and Siegel (1986).[1] The Poisson arrival process, as in search-theoretic models, stands for illiquidity of option exercise opportunities. Most notably, I derive the closed-form solution in the proposed illiquid model. The obtained solution exhibits intermediate characteristics between the NPV and ROV solutions. Indeed, the option value of waiting decreases due to illiquidity, and hence, the investment threshold decreases. This implies that in the presence of illiquidity, the firm exercises an investment option more eagerly than in the ROV model. Although Boyle and Guthrie (2003) document similar results from capital market friction, they do not derive any analytical solution due to the model complexity.

Furthermore, I also examined a case in which two types of projects arrive randomly. Although I cannot derive a closed-form solution in this case, I obtain the analytical solution. When the profitability gap between the two types, liquidity, and the probability of the good-type appearance increase, the firm is more likely to wait for a good-type project and forgo a bad-type project. When these factors are low, the firm accepts only a good-type project for intermediate levels of the state variable and accepts even a bad-type project for sufficiently high levels of the state variable. The impact of illiquidity on the bad-type investment threshold is much greater than on the good-type investment threshold. This result is also numerically verified for a case with a continuum of project types.

I prove that the solution of the illiquid model converges to that of the ROV model for higher liquidity and that of the NPV model for lower liquidity. I also reveal the relationship of the illiquid model to the regime-switching model of Hackbarth, Miao, and Morellec (2006); indeed, the limiting solution of the regime-switching model concurs with the solution of the illiquid model. Thus, in terms of illiquidity of real option exercise opportunities, this paper fills the gaps in the NPV, ROV, and regime-switching models. The proposed models and solutions can potentially play a significant role as a new framework to study real options with illiquidity of option exercise opportunities. They can be applied to various problems, including but not limited to problems of merger and acquisition timing with illiquidity of searching and matching, illiquid asset sales and liquidation timing, and investment timing with illiquidity of fundraising.

The remainder of this paper is organized as follows. Section 2 introduces the model

[1] Although I describe the results of the investment timing model in full details, the same analysis can be applied to asset sales and liquidation models (cf. Section 4.3).

setup. Then, Sections 3.1 and 3.2 explain the NPV and ROV models. Sections 3.3 and 3.4 show the main results of the models with illiquidity of option exercise opportunities. Section 4 extends these models in several ways, and Section 5 concludes the paper.

2 Setup

Consider a firm that has an option to invest in a project only when an investment opportunity arrives. Assume that the project arrival process follows a Poisson process with the arrival rate $\lambda(>0)$. If the firm rejects a project, the firm has to wait until another project arrives. The firm cannot invest in more than one project.[2]

Project i requires the initial investment cost I_i and perpetually generates cash flow $a_iX(t)$. The random shock $X(t)$ follows a geometric Brownian motion

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}B(t)\ (t > 0), \quad X(0) = x,$$

where $B(t)$ denotes the standard Brownian motion defined in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ and $\mu, \sigma(> 0)$ and $x(> 0)$ are constants. Throughout the paper, for the model tractability, I assume that $B(t)$ is independent of the Poisson process. A positive constant r denotes the discount rate, and for convergence I assume that $r > \mu$.

I consider two types of projects, i.e., $i = G$ (Good) and $i = B$ (Bad), where $0 < a_B < a_G$ and $0 < I_G \leq I_B$. Define the expected payoff of project i by $\pi_i(x) = a_ix/(r-\mu) - I_i$. In terms of profitability, a project G dominates a project B because $\pi_G(x) > \pi_B(x)$ holds for all $x > 0$.[3] Section 3.3 examines a case in which the same type of projects arrive. Section 3.4 examines a case in which a project type follows an independent draw with $i = G$ at probability $q \in (0,1)$ and $i = B$ at probability $1-q$.

The model incorporates illiquidity of investment opportunities in the ROV model of McDonald and Siegel (1986). As in search-theoretic models (e.g., Diamond (1982) and Duffie, Garleanu, and Pedersen (2005)), illiquidity is captured by the Poisson arrival process. The illiquid model is relevant to real options involving transactions of illiquid assets, such as real estate, technology, patents, and businesses. It can also represent illiquidity of fundraising. Several papers (e.g., Baldwin and Meyer (1979) and Baldwin (1982)), prior to development of the real options literature, investigate investment timing

[2]Investment projects are mutually exclusive or the firm chooses one project under financial constraints.

[3]Unlike this paper, Décamps, Mariotti, and Villeneuve (2006) consider two mutually exclusive projects with different scales, i.e., $0 < a_B < a_G$ and $0 < I_B < I_G$. They solved the optimal investment timing and sizing problem under the standard real options assumption that investment is possible at an arbitrary time.

under the assumption that investment opportunities arrive according to a Poisson process, but these models do not include a continuous-time state process, i.e., $X(t)$. Section 4.2 shows the relationship of the illiquid model to the regime-switching model of Hackbarth, Miao, and Morellec (2006). I can apply the same analysis used in this paper to put-type real options (cf. Section 4.3).

3 Model Solutions

3.1 The NPV model

As a benchmark, this subsection briefly explains the NPV framework. Suppose that a project of type i, which is fixed at G or B, is feasible at the initial time.[4] Consider the firm that follows the NPV rule, i.e., the firm undertakes a project if and only if $\pi_i(x) \geq 0$ at the initial time.

The firm value with a project i, denoted by $NPV_i(x)$, is given by

$$NPV_i(x) = \max\{\pi_i(x), 0\} \quad (i = G, B). \tag{1}$$

The investment threshold in the NPV model is given by

$$x_i^{NPV} = \frac{(r-\mu)I_i}{a_i} \quad (i = G, B). \tag{2}$$

The firm invests in a project i if $x \geq x_i^{NPV}$. Otherwise, it never invests in a project. The NPV rule is optimal if the firm has no option to defer the investment timing.

3.2 The ROV model

As another benchmark, this subsection explains the ROV model, which was first studied in McDonald and Siegel (1986). Suppose that a project of type i, which is fixed G or B, is feasible at any time.[5] Consider the firm that follows the ROV rule, i.e., the firm undertakes a project at the optimal time.

For future uses, I define, for $y > 0$,

$$\beta_y = 0.5 - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - 0.5\right)^2 + \frac{2(r+y)}{\sigma^2}} (> 1), \tag{3}$$

$$\gamma_y = 0.5 - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - 0.5\right)^2 + \frac{2(r+y)}{\sigma^2}} (< 0). \tag{4}$$

[4] The firm focuses only on project G when both types of projects are feasible at the initial time.

[5] The firm focuses only on project G when both types of projects are feasible at an arbitrary time.

For notational simplicity, I denote $\beta = \beta(0)$ and $\gamma = \gamma(0)$. In the standard manner (e.g., Dixit and Pindyck (1994)), I can derive the firm value with a project i as

$$ROV_i(x) = \begin{cases} \pi_i(x_i^{ROV}) \left(\dfrac{x}{x_i^{ROV}} \right)^{\beta} & (x < x_i^{ROV}), \\ \pi_i(x) & (x \geq x_i^{ROV}), \end{cases} \quad (i = G, B), \tag{5}$$

where the investment threshold is given by

$$x_i^{ROV} = \frac{\beta x_i^{NPV}}{\beta - 1} (< x_i^{NPV}) \quad (i = G, B). \tag{6}$$

For $x \geq x_i^{ROV}$, the firm invests in a project i at the initial time. In this case, $ROV_i(x) = NPV_i(x)$ holds, which means the ROV method adds no value to the NPV method. For $x < x_i^{ROV}$, the firm invests in a project i as soon as the state variable $X(t)$ hits x_i^{ROV}. The upper equation in (5) stands for the value of deferring investment. Because of the option value of waiting, $ROV_i(x) > NPV_i(x)$ holds for $x < x_i^{ROV}$.

3.3 The same type of projects arriving randomly

This subsection examines the baseline model in which the same type of projects arrive randomly. To be more precise, I assume that projects of type i, which is fixed at G or B, arrive according to a Poisson process with the arrival rate λ. I denote by $V_i(x)$ the firm value when a project i is not feasible and by $\tilde{V}_i(x)$ the firm value when a project i arrives. I have $\tilde{V}_i(x) = \max\{\pi_i(x), V_i(x)\}$ because the firm value $\tilde{V}_i(x)$ becomes $V_i(x)$ when a feasible project i is forgone.

I can show that the value function $\tilde{V}_i(x)$ is continuous and that the optimal policy is a threshold policy. For the proof, see Appendix A. Then, the stopping region $S_i^* = \{x \in \mathbb{R}_+ \mid \tilde{V}_i(x) = \pi_i(x)\}$ is expressed as

$$S_i^* = \{x \in \mathbb{R}_+ \mid x \geq x_i^*\}, \tag{7}$$

where $x_i^* = \min\{x \in \mathbb{R}_+ \mid \tilde{V}_i(x) = \pi_i(x)\}$, and $\tilde{V}_i(x)$ is expressed as

$$\tilde{V}_i(x) = \begin{cases} V_i(x) & (x < x_i^*), \\ \pi_i(x) & (x \geq x_i^*). \end{cases} \tag{8}$$

Because of $NPV_i(x) \leq \tilde{V}_i(x) \leq ROV_i(x)$, the investment threshold x_i^* exists between x_i^{NPV} and x_i^{ROV}. Because of Theorem 4.4.9 in Karatzas and Shreve (1998), for the threshold policy, the expectation $V_i(x)$ is a piecewise C^2 function and satisfies ordinary differential equations (ODEs):

$$\mu x V_i'(x) + 0.5\sigma^2 x^2 V_i''(x) = rV_i(x) \quad (0 < x < x_i^*), \tag{9}$$

$$\mu x V_i'(x) + 0.5\sigma^2 x^2 V_i''(x) + \lambda(\pi_i(x) - V_i(x)) = rV_i(x) \quad (x > x_i^*), \tag{10}$$

where $\lambda(\pi_i(x) - V_i(x))$ corresponds to the fact that $V_i(x)$ changes to $\pi_i(x)$ with probability $\lambda \mathrm{d}t$ in infinitesimal time interval $\mathrm{d}t$. The piecewise C^2 property means that $V_i(x)$ is continuously differentiable at x_i^* (see Theorem 4.4.9 in Karatzas and Shreve (1998)). By solving (9) and (10) with the boundary conditions which will be specified in the proof, I have the following proposition.

Proposition 1

$$V_i(x) = \begin{cases} A_{1,i} x^{\beta} & (x < x_i^*), \\ \dfrac{\lambda a_i x}{(r-\mu)(r+\lambda-\mu)} - \dfrac{\lambda I_i}{r+\lambda} + A_{2,i} x^{\gamma_\lambda} & (x \geq x_i^*), \end{cases} \tag{11}$$

where the investment threshold x_i^* and coefficients $A_{1,i}, A_{2,i}$ are

$$x_i^* = \frac{(r+\lambda-\mu)((\beta-\gamma_\lambda)r + \beta\lambda)x_i^{NPV}}{(r+\lambda)((\beta-\gamma_\lambda)(r-\mu)+(\beta-1)\lambda)}, \tag{12}$$

$$A_{1,i} = \frac{\pi_i(x_i^*)}{x_i^{*\beta}}, \tag{13}$$

$$A_{2,i} = \frac{1}{x_i^{*\gamma_\lambda}} \left(\frac{a_i x_i^*}{r+\lambda-\mu} - \frac{r I_i}{r+\lambda} \right). \tag{14}$$

Proof. I have the boundary conditions

$$\lim_{x \to 0} V_i(x) = 0 \tag{15}$$

$$\lim_{x \to \infty} \frac{V_i(x)}{\pi_i(x)} < \infty \tag{16}$$

$$V_i(x^*) = \pi_i(x^*) \tag{17}$$

along with the continuous differentiability of $V_i(x)$ at x_i^*. Conditions (15) and (16) are trivial, while condition (17) follows from (8) and the continuity of $\tilde{V}_i(x)$. The continuous differentiability of $V_i(x)$ at x_i^* follows from Theorem 4.4.9 in Karatzas and Shreve (1998)).[6]

A general solution to ODE (9) with (15) is expressed as $A_{1,i}x^{\beta}$, where $A_{1,i}$ is an unknown coefficient. Then, by (16), I have the expression (13). A general solution to ODE (10) with (16) is expressed as

$$\frac{\lambda a_i x}{(r-\mu)(r+\lambda-\mu)} - \frac{\lambda I_i}{r+\lambda} + A_{2,i} x^{\gamma_\lambda},$$

[6] This is not the smooth pasting (i.e., optimality) condition in Dixit and Pindyck (1994), but it is the same condition as the piecewise C^2 property in the regime-switching models in Guo, Miao, and Morellec (2005) and Hackbarth, Miao, and Morellec (2006). I do not have to impose the smooth pasting condition and verify the optimality of the solution. Instead, I can directly calculate the expectation $V_i(x)$ by condition (17) which stems from the optimality of x_i^*.

where $A_{2,i}$ is an unknown coefficient. Then, by (17), I have the expression (14). By the continuous differentiability of $V_i(x)$ at x_i^*, I have

$$\beta A_1 x_i^{*\beta-1} = \frac{\lambda a_i}{(r-\mu)(r+\lambda-\mu)} + \gamma_\lambda A_2 x_i^{*\gamma_\lambda-1}. \tag{18}$$

By substituting (13) and (14) into (18), I have

$$\begin{aligned} \beta\pi_i(x_i^*) &= \frac{\lambda a_i x_i^*}{(r-\mu)(r+\lambda-\mu)} + \gamma_\lambda\left(\frac{a_i x_i^*}{r+\lambda-\mu} - \frac{rI_i}{r+\lambda}\right) \\ x_i^* &= \frac{(r+\lambda-\mu)((\beta-\gamma_\lambda)r+\beta\lambda)x_i^{NPV}}{(r+\lambda)((\beta-\gamma_\lambda)(r-\mu)+(\beta-1)\lambda)}. \end{aligned}$$

The proof is complete. □

The firm's optimal policy is to invest in a project i only if it arrives at time t satisfying $X(t) \geq x_i^*$. Unlike in the standard real options model, the investment time is later than the first hitting time to the investment threshold x_i^*. Indeed, the firm cannot invest at the first hitting time when a project is not feasible.[7] Lower liquidity λ increases the gap between the first hitting time and the investment time.

The upper equation in (11) stands for the value of waiting when the option value is higher than the investment payoff, i.e, $V_i(x) > \pi_i(x)$. The lower equation in (11) stands for the value of waiting when the option value is lower than the investment payoff, i.e, $V_i(x) \leq \pi_i(x)$. Then, the firm wishes to immediately exercise the investment option, but it cannot do so because of illiquidity of exercise opportunities. To be more precise, the lower equation in (11) can be decomposed as follows. The first and second terms of the lower equation stand for the expected payoff of investing as soon as a project i arrives. The last term $A_{2,i}x^{\gamma_\lambda}$ stands for the value of the option to forgo a project i when it arrives at time t satisfying $X(t) < x_i^*$.

Notably, I have the closed-form solution in Proposition 1. Indeed, the investment threshold x_i^* is derived in the closed-form expression (12). In a more generalized model (i.e., $X(t)$ following a geometric Lévy process with one-sided jumps), Perez and Yamazaki (2018) show an analytical solution, but they do not derive any closed-form solution. Similarly, Hackbarth, Miao, and Morellec (2006) show an analytical solution in the general regime-switching model, but they do not derive any closed-form solution.[8] Sections 4.1–4.3 also present closed-form solutions in several extended models. By virtue of tractability, the closed-form solutions obtained in this paper will be a framework to study real options with illiquidity of option exercise opportunities.

[7] Mathematically, the probability of the Poisson arrival at the first hitting time is equal to zero.

[8] In Section 4.2, I derive the closed-form solution of the regime-switching model where investment is feasible in one of the two regimes and reveal the relationship of the illiquid model with the regime-switching model.

Next, I examine the effects of liquidity λ on the investment threshold and firm values. I will show the following results in the limiting cases.

Proposition 2 *$V_i(x), \tilde{V}_i(x)$, and x_i^* monotonically increase in λ.*

$$\lim_{\lambda\to 0} V_i(x) = 0,\ \lim_{\lambda\to 0} \tilde{V}_i(x) = NPV_i(x),\ \lim_{\lambda\to 0} x_i^* = x_i^{NPV},$$
$$\lim_{\lambda\to\infty} V_i(x) = \lim_{\lambda\to\infty} \tilde{V}_i(x) = ROV_i(x),\ \lim_{\lambda\to\infty} x_i^* = x_i^{ROV}.$$

Proof. By definition of the problem, $V_i(x)$ and $\tilde{V}_i(x)$ monotonically increase in the arrival rate λ. By the monotonicity of $\tilde{V}_i(x)$ and $x_i^* = \min\{x \in \mathbb{R}_+ \mid \tilde{V}_i(x) = \pi_i(x)\}$, I have the monotonicity of x_i^*.

By $\lambda \to 0$ in (12), I have

$$\lim_{\lambda\to 0} x_i^* = x_i^{NPV}. \tag{19}$$

By substituting (19) into (13) and (14), I have $\lim_{\lambda\to 0} A_{1,i} = \lim_{\lambda\to 0} A_{2,i} = 0$. Then, $\lim_{\lambda\to 0} V_i(x) = 0$ holds. By (8), I also have $\lim_{\lambda\to 0} \tilde{V}_i(x) = NPV_i(x)$.

Now I consider the limiting case of $\lambda \to \infty$. Note that $\lim_{\lambda\to\infty} \gamma_\lambda/\lambda = 0$. By this property and (12), I have

$$\lim_{\lambda\to\infty} x_i^* = x_i^{ROV}. \tag{20}$$

By substituting (20) into (13), (14), and (11), I have $\lim_{\lambda\to\infty} V_i(x) = ROV_i(x)$. By (8), I also have $\lim_{\lambda\to\infty} \tilde{V}_i(x) = ROV_i(x)$. The proof is completed. □

Lower liquidity of investment opportunities decreases the value of waiting and the investment threshold x_i^* below x_i^{ROV}. In other words, the firm accepts an investment opportunity more eagerly than in the ROV model because the firm has to wait for another project arrival if it forgoes an investment opportunity. This is similar to Boyle and Guthrie (2003)'s result that cash shortfall risk decreases the value of waiting and the hurdle rate for investment, although they do not derive any analytical solution due to the model complexity.

Proposition 2 shows that the NPV and ROV solutions are obtained in the limiting cases of $\lambda \to 0$ and $\lambda \to \infty$, respectively.[9] Indeed, as λ decreases (increases), $x_i^*, V_i(x)$, and $\tilde{V}_i(x)$ approach the NPV (ROV) solutions. The baseline model results fill the gaps between the two extreme cases (i.e., the NPV and ROV models) by showing the intermediate solution in the closed form. Indeed, it is more general and plausible to assume that the firm can delay

[9] Although Perez and Yamazaki (2018) numerically check the limiting cases (for a geometric Lévy process with one-sided jumps $X(t)$), they do not analytically prove the limiting cases.

the investment time but cannot choose an arbitrary investment time. A Bermudan option, where the holder can exercise the option only on certain predefined dates, also captures an intermediate case between the NPV and ROV models. In this case, however, the solution cannot be analytically derived but can be numerically computed in the corresponding discrete-time model. In contrast, by virtue of randomizing exercise dates, I can derive the closed-form solution in the baseline model.

Lastly, I check the impacts of illiquidity on the investment thresholds and firm values in numerical examples. The baseline parameter values are set in Table 1. We now consider the case of $i = G$. Figure 1 shows the firm value function $V_G(x)$ along with the benchmark values $ROV_G(x)$ and $NPV_G(x)$ for x close to the investment threshold $x_G^* = 0.8088$. Note that $NPV_G(x) = \pi_G(x)$ holds in the depicted region. As shown in Proposition 1, $V_G(x)$ crosses $\pi_G(x)$ at the investment threshold $x_G^* = 0.8088$. As shown in Proposition 1, the investment threshold $x_G^* = 0.8088$ lies between $x_G^{NPV} = 0.2$ and $x_G^{ROV} = 0.9531$.

Figure 2 shows $V_G(x), ROV_G(x), x_G^*$, and x_G^{ROV} along with the liquidity cost $LC = (ROV_G(x) - V_G(x))/ROV_G(x)$ and the threshold decrease $TD = (x_G^{ROV} - x_G^*)/x_G^{ROV}$ for varying levels of $1/\lambda$. Note that $1/\lambda$ is equal to the expected time interval (years) between two projects. The firm values are computed for the initial value $x = 0.2$, where $NPV_G(x) = 0$ holds. As shown in Proposition 2, both $V_G(x)$ and x_G^* monotonically decrease in $1/\lambda$. More interestingly, I can see from the figure that the impact of illiquidity is greater on the investment threshold than on the firm value. For instance, I have $LC = 0.005$ and $TD = 0.1515$ for $\lambda = 1$. Note that LC does not depends on the initial value x for $x \leq x_G^*$ and that LC increases to $(r-\mu)/(r+\lambda-\mu)$ for $x \to \infty$. As expected from the expression $(r-\mu)/(r+\lambda-\mu)$, LC can be significantly high for plausible levels of $1/\lambda$ when $r - \mu$ is quite large (say, negative μ[10]). By Proposition 2, I can easily show that $\lim_{\lambda\to 0} LC = 1, \lim_{\lambda\to 0} TD = 1/\beta$, and $\lim_{\lambda\to\infty} LC = \lim_{\lambda\to\infty} TD = 0$.

Figure 3 shows $V_G(x), ROV_G(x), x_G^*, x_G^{ROV}, LC$, and TD for varying levels of σ. The parameter values other than σ are set in Table 1. It is well known that $ROV_G(x)$ and x_G^{ROV} monotonically increase in σ because a higher σ increases the value of deferring investment (e.g., Dixit and Pindyck (1994)). In the NPV model, volatility has no effect on the firm value and investment threshold. Taking into account that the baseline model is intermediate between the ROV and NPV models, I can conjecture that the volatility effects in the baseline model are weaker than in the ROV model. Figure 3 verifies this conjecture. Indeed, both $V_G(x)$ and x_G^* monotonically increase in σ, but the increases

[10]If I set $\mu = -0.05$ in Table 1, I have $LC = 0.1428$ and $TD = 0.1079$.

are moderate compared to $ROV_G(x)$ and x_G^{ROV} (see the top panels). As a result, the impacts of illiquidity, i.e., LC and TD, monotonically increase in σ (see the bottom panels). By (12), I can easily show that $\lim_{\sigma\to 0} x_i^* = x_i^{NPV}(r+\lambda-\mu)/(r+\lambda)$ for $\mu > 0$ and $\lim_{\sigma\to 0} x_i^* = x_i^{NPV}$ for $\mu \leq 0$. I also have $\lim_{\sigma\to\infty} x_i^* = x_i^{NPV}(r+\lambda-\mu)/(r-\mu)$, which is sharply contrasted with the unbounded threshold $\lim_{\sigma\to\infty} x_i^{ROV} = \infty$ in the ROV model. Then, I can show that $\lim_{\sigma\to 0} LC = 1 - ((r+\lambda)/(r+\lambda-\mu))^{r/\mu}\lambda/(r+\lambda)$, $\lim_{\sigma\to 0} TD = \mu/(r+\lambda)$ for $\mu > 0$ and $\lim_{\sigma\to 0} TD = 0$ for $\mu \leq 0$. I can also show that $\lim_{\sigma\to\infty} LC = (r-\mu)/(r+\lambda-\mu)$, $\lim_{\sigma\to\infty} TD = 1$.

3.4 Two types of projects arriving randomly

Throughout this paper, I investigate illiquidity of investment opportunities. Although I focus on exclusive investment projects, projects might be different from each others. For instance of a merger and acquisition option with illiquidity of searching and matching, the acquiring firm might meet different targets, where the synergies in the merger and acquisition vary over targets. It is important to see how the baseline results in Section 3.3 change with multiple project types. To do so, this section extends the baseline model to a case in which two types of projects arrive randomly.[11] To be more precise, I assume that projects arrive following a Poisson process with the arrival rate λ and that project type i equals G at probability q and equals B at probability $1-q$ by an independent draw. The firm can choose between acceptance and rejection after it observes project type i. I denote by $W(x)$ the firm value when no project is feasible and by $\tilde{W}_i(x)$ $(i = G, B)$ the firm values when a project i arrives. I have $\tilde{W}_i(x) = \max\{\pi_i(x), W(x)\}$ $(i = G, B)$ because the firm value $\tilde{W}_i(x)$ becomes $W(x)$ when a feasible project i is forgone.

As in Section 3.3, I can show that $\tilde{W}_i(x)$ $(i = G, B)$ are continuous and the optimal policy is a threshold policy. For the proof, refer to Appendix B. To be more precise, the stopping regions $S_i^{**} = \{x \in \mathbb{R}_+ \mid \tilde{W}_i(x) = \pi_i(x)\}$ $(i = G, B)$ are expressed as

$$S_i^{**} = \{x \in \mathbb{R}_+ \mid x \geq x_i^{**}\} \quad (i = G, B) \tag{21}$$

Because of $NPV_G(x) \leq \tilde{W}_G(x) \leq ROV_G(x)$, the good-type investment threshold x_G^{**} exists between x_G^{NPV} and x_G^{ROV}. Because of $\tilde{W}_B(x) \not\leq ROV_B(x)$, the bad-type threshold $x_B^{**}(\geq \max(x_G^{**}, x_B^{NPV}))$ can be infinite, i.e., S_B^{**} can be empty. I will later clarify the

[11] I can similarly derive the solution in the illiquid model with a finite number of project types. Section 4.4 examines a continuum of project types. In Baldwin and Meyer (1979) and Baldwin (1982), project types has a continuous distribution, although the models do not include the state process, i.e., $X(t)$.

condition under which S_B^{**} is empty. The value functions $\tilde{W}_i(x)$ $(i = G, B)$ are expressed as

$$\tilde{W}_i(x) = \begin{cases} W(x) & (x < x_i^{**}), \\ \pi_i(x) & (x \geq x_i^{**}) \end{cases} \quad (i = G, B). \tag{22}$$

If S_B^{**} is empty, I can derive $W(x)$ in the same manner as in Proposition 1. Suppose that x_B^{**} is finite. As in Section 3.3, for the threshold policy, the expectation $W(x)$ is a piecewise C^2 function and satisfies ODEs:

$$\mu x W'(x) + 0.5\sigma^2 x^2 W''(x) = rW(x) \quad (0 < x < x_G^{**}), \tag{23}$$

$$\mu x W'(x) + 0.5\sigma^2 x^2 W''(x) + \lambda q(\pi_G(x) - W(x)) = rW(x) \quad (x_G^{**} < x < x_B^{**}), \tag{24}$$

$$\mu x W'(x) + 0.5\sigma^2 x^2 W''(x) + \lambda(\bar{\pi}(x) - W(x)) = rW(x) \quad (x > x_B^{**}), \tag{25}$$

where I define $\bar{\pi}(x) = q\pi_G(x) + (1-q)\pi_B(x)$. In (24), $\lambda q(\pi_G(x) - W(x))$ corresponds to the fact that $W(x)$ changes to $\pi_G(x)$ with probability $\lambda q \mathrm{d}t$ in infinitesimal time interval $\mathrm{d}t$, and in (25), $\lambda(\bar{\pi}(x) - W(x))$ corresponds to the fact that $W(x)$ changes to $\bar{\pi}(x)$ with probability $\lambda \mathrm{d}t$ in $\mathrm{d}t$. As in $V_i(x)$ in Section 3.3, $W(x)$ is continuously differentiable at thresholds x_i^{**} $(i = G, B)$ because of Theorem 4.4.9 in Karatzas and Shreve (1998). By solving (23)–(25) with the boundary conditions which will be specified in the proof, I have the following proposition. In the proposition, I define $\bar{a} = qa_G + (1-q)a_B$ and $\bar{I} = qI_G + (1-q)I_B$, and I denote (8), (11), and (12) by $\tilde{V_G}(x;\lambda), V_G(x;\lambda)$, and $x_G^*(\lambda)$, respectively, to indicate argument λ.

Proposition 3 *Suppose that*

$$\frac{r-\mu}{\lambda q} + 1 \leq \frac{a_G}{a_B}. \tag{26}$$

The firm has no possibility of investing in a project B. The firm value $W(x)$ is given by

$$W(x) = V_G(x; \lambda q).$$

The investment thresholds x_i^{**} $(i = G, B)$ are given by

$$x_G^{**} = x_G^*(\lambda q), \ x_B^{**} = \infty.$$

Suppose that (26) does not hold. Assume that there exists a unique solution (x_G, x_B) satisfying $x_G^{NPV} \leq x_G < x_B$ and

$$\frac{\lambda q a_G x_G}{(r-\mu)(r+\lambda q-\mu)} + \beta_{\lambda q} B_2 x_G^{\beta_{\lambda q}} + \gamma_{\lambda q} B_3 x_G^{\gamma_{\lambda q}} = \beta \pi_G(x_G), \tag{27}$$

$$\frac{\lambda q a_G x_B}{(r-\mu)(r+\lambda q-\mu)} + \beta_{\lambda q} B_2 x_B^{\beta_{\lambda q}} + \gamma_{\lambda q} B_3 x_B^{\gamma_{\lambda q}} = \frac{\lambda \bar{a} x_B}{(r-\mu)(r+\lambda-\mu)} + \gamma_\lambda C_3, \tag{28}$$

where B_i $(i = 1, 2, \ldots, 4)$ *and* C_i $(i = 1, 2, 3)$ *are defined by*

$$\begin{aligned}
B_1 &= \frac{\pi_G(x_G)}{x_G^{\beta}}, \\
B_2 &= \frac{x_G^{\gamma_{\lambda q}} C_2 - x_B^{\gamma_{\lambda q}} C_1}{x_B^{\beta_{\lambda q}} x_G^{\gamma_{\lambda q}} - x_G^{\beta_{\lambda q}} x_B^{\gamma_{\lambda q}}}, \\
B_3 &= \frac{x_B^{\beta_{\lambda q}} C_1 - x_G^{\beta_{\lambda q}} C_2}{x_B^{\beta_{\lambda q}} x_G^{\gamma_{\lambda q}} - x_G^{\beta_{\lambda q}} x_B^{\gamma_{\lambda q}}}, \\
B_4 &= \frac{C_3}{x_B^{\gamma_\lambda}}, \\
C_1 &= \frac{a_G x_G}{r + \lambda q - \mu} - \frac{r I_G}{r + \lambda q}, \\
C_2 &= \pi_B(x_B) - \frac{\lambda q a_G x_B}{(r - \mu)(r + \lambda q - \mu)} + \frac{\lambda q I_G}{r + \lambda q}, \\
C_3 &= \pi_B(x_B) - \frac{\lambda \bar{a} x_B}{(r - \mu)(r + \lambda - \mu)} + \frac{\lambda \bar{I}}{r + \lambda}.
\end{aligned}$$

The investment thresholds x_i^{**} $(i = G, B)$ *are the above solution, i.e.,*

$$x_G^{**} = x_G, \ x_B^{**} = x_B.$$

The firm value $W(x)$ *is given by*

$$W(x) = \begin{cases} B_1 x^{\beta} & (x < x_G^{**}), \\ \dfrac{\lambda q a_G x}{(r - \mu)(r + \lambda q - \mu)} - \dfrac{\lambda q I_G}{r + \lambda q} + B_2 x^{\beta_{\lambda q}} + B_3 x^{\gamma_{\lambda q}} & (x \in [x_G^{**}, x_B^{**})), \\ \dfrac{\lambda \bar{a} x}{(r - \mu)(r + \lambda - \mu)} - \dfrac{\lambda \bar{I}}{r + \lambda} + B_4 x^{\gamma_\lambda} & (x \geq x_B^{**}). \end{cases} \tag{29}$$

Proof. Assume that (26) holds. In this case, I can show that $V_G(x; \lambda q) > \pi_B(x)$ $(x \in \mathbb{R}_+)$ as follows. For $x < x_G^*(\lambda q)$, I have

$$V_G(x; \lambda q) = \tilde{V_G}(x; \lambda q) > \pi_G(x) \geq \pi_B(x). \tag{30}$$

By $A_{2,i} > 0$ in (11), for $x \geq x_G^*(\lambda q)$, I have

$$\begin{aligned}
V_G(x; \lambda q) &> \frac{\lambda q a_G x}{(r - \mu)(r + \lambda q - \mu)} - \frac{\lambda q I_G}{r + \lambda q} \\
&\geq \frac{a_B x}{r - \mu} - \frac{\lambda q I_G}{r + \lambda q} \qquad (31) \\
&> \pi_B(x), \qquad (32)
\end{aligned}$$

where (31) follows from (26). By (30) and (32), I have $V_G(x; \lambda q) > \pi_B(x)$ $(x \in \mathbb{R}_+)$. Then, I have

$$\tilde{W_B}(x) \geq W(x) \geq V_G(x; \lambda q) > \pi_B(x) \ (x \in \mathbb{R}_+),$$

which implies that S_B^{**} is empty. The firm has no possibility of investing in a project B, and hence, the problem is reduced to the problem when projects G arrive according to a Poisson process with the arrival rate λq. By Proposition 1, I have $W(x) = V_G(x; \lambda q)$ and $x_G^{**} = x_G^*(\lambda q)$.

Next, assume that (26) does not hold. I can readily see that $V_G(x'; \lambda q) < \pi_B(x')$ for a sufficiently large x'. This implies that S_B^{**} is not empty. Indeed, if S_B^{**} is empty, I have

$$\tilde{W}_B(x') = \max\{\pi_B(x'), W(x')\} = W(x') = V_G(x'; \lambda q),$$

which contradicts the inequality $V_G(x'; \lambda q) < \pi_B(x')$. Then, two finite thresholds x_i^{**} $(i = G, B)$ exist, and hence, I have ODEs (23)–(25). I have the boundary conditions

$$\lim_{x \to 0} W(x) = 0 \tag{33}$$

$$\lim_{x \to \infty} \frac{W(x)}{\pi_B(x)} < \infty \tag{34}$$

$$W(x_i^{**}) = \pi_i(x_i^{**}) \quad (i = G, B), \tag{35}$$

along with the continuous differentiability of the continuous differentiability of $W(x)$ at x_i^{**} $(i = G, B)$. Conditions (33) and (34) are trivial, and condition (35) follows from (22) and the continuity of $\tilde{W}_i(x)$ $(i = G, B)$. The continuous differentiability of $W(x)$ at x_i^{**} $(i = G, B)$ follows from Theorem 4.4.9 in Karatzas and Shreve (1998). ODE (23) with boundary conditions (33) and (35), ODE (24) with boundary condition (35), and ODE (25) with boundary conditions (34) and (35) lead to the first, second, and third rows in (29), respectively. Because of the continuous differentiability of $W(x)$ at x_i^{**} $(i = G, B)$, the investment thresholds x_i^{**} $(i = G, B)$ satisfy (27) and (28). The proof is complete. □

The solution varies according to whether (26) holds or not. In the former case, the firm always ignores a project B, and hence, the solution is the same as in Proposition 1 with $i = G$ and the arrival rate λq. I can see from (26) that higher $a_G/a_B, \lambda, q$, and lower $r - \mu$ are likely to lead this case.

When (26) does not hold, the firm takes the following policy. The firm forgoes any project when it arrives at time t satisfying $X(t) < x_G^{**}$. The firm undertakes a project G if it arrives at time t satisfying $X(t) \geq x_G^{**}$. Notably, the firm accepts only a project G for $X(t) \in [x_G^{**}, x_B^{**})$. The firm undertakes even a project B if it arrives at time t satisfying $X(t) \geq x_B^{**}$. As explained after Proposition 1, the investment time is later than the first hitting time to the threshold. The paths of $X(t)$ and the Poisson process determine whether the firm invests in a project G or B. This unpredictability is a key difference from the ROV model in which the firm invests in a project G at the first hitting time. Due to illiquidity of investment opportunities, unlike in the ROV model, the firm can potentially invest in a project B.

Although I cannot derive closed-form expressions of x_G^{**} and x_B^{**}, I have the analytical expression of $W(x)$ in Proposition 3. In (29), the first row stands for the value of waiting when the option value is higher than the good-type investment payoff, i.e, $W(x) > \pi_G(x)$. The second row stands for the value of waiting when the option value is between the bad- and good-type investment payoffs, i.e, $\pi_B(x) < W(x) \leq \pi_G(x)$. Then, the firm wishes to immediately invest in a project G, but it cannot do so because of illiquidity of investment opportunities. I can decompose the second row in (29) as follows. The first and second terms represent the expected payoff of investing whenever a project G arrives. $B_2x^{\beta_{\lambda q}} + B_3x^{\gamma_{\lambda q}}$ represents the value of the option to revise this investment policy when $X(t)$ passes either x_G^{**} or x_B^{**} before the arrival of a project G. The last row in (29) stands for the value of waiting when the option value is lower than the bad-type investment payoff, i.e, $W(x) \leq \pi_B(x)$. The first and second terms of this equation represent the expected payoff of investing whenever a project arrives regardless of its type. The last term $B_4x^{\gamma_\lambda}$ represents the value of the option to revise this investment policy when $X(t)$ passes x_B^{**} before the arrival of a project.

Next, I examine the effects of liquidity λ on the investment thresholds and firm values. I will show the following results in the limiting cases under the assumptions in Proposition 3.

Proposition 4 *$W(x), \tilde{W}_i(x)$, and x_i^{**} monotonically increase in λ.*

$$\lim_{\lambda\to 0} W(x) = 0,\ \lim_{\lambda\to 0} \tilde{W}_i(x) = NPV_i(x),\ \lim_{\lambda\to 0} x_i^{**} = x_i^{NPV}\ \ (i = G, B).$$

When $\lambda \to \infty$, (26) is satisfied, and

$$\lim_{\lambda\to\infty} W(x) = \lim_{\lambda\to\infty} \tilde{W}_G(x) = \lim_{\lambda\to\infty} \tilde{W}_B(x) = ROV_G(x),\ \lim_{\lambda\to\infty} x_G^{**} = x_G^{ROV},\ \lim_{\lambda\to\infty} x_B^{**} = \infty.$$

Proof. By definition of the problem, $W(x)$ and $\tilde{W}_i(x)$ monotonically increase in the arrival rate λ. Then, the monotonicity of x_i^{**} follows from $x_i^{**} = \min\{x \in \mathbb{R}_+ \mid \tilde{W}_i(x) = \pi_i(x)\}$. First, consider the limiting case of $\lambda \to 0$. (26) is not satisfied as $\lambda \to 0$. In the limiting case, I have

$$B_2{x_G^{**}}^{\beta} + B_3{x_G^{**}}^{\gamma} = \pi_G(x_G^{**}) \tag{36}$$

by $W(x_G^{**}) = \pi_G(x_G^{**})$ (see (29)) and

$$\beta B_2{x_G^{**}}^{\beta} + \gamma B_3{x_G^{**}}^{\gamma} = \beta\pi_G(x_G^{**}) \tag{37}$$

by (27). By (36) and (37), I have $(\beta-\gamma)B_3 x_G^{**\gamma}=0$. Because of $x_G^{**} \geq x_G^{NPV} > 0$, I have $B_3 = 0$. Similarly, I have

$$B_2 x_B^{**\beta} + B_3 x_B^{**\gamma} = \pi_B(x_B^{**}) \tag{38}$$

by $W(x_B^{**}) = \pi_B(x_B^{**})$ (see (29)) and

$$\beta B_2 x_B^{**\beta} + \gamma B_3 x_B^{**\gamma} = \gamma \pi_B(x_B^{**}) \tag{39}$$

by (28). By (38) and (39), I have $(\beta-\gamma)B_2 x_B^{**\beta}=0$. Because of $x_B^{**} > x_G^{**} > 0$, I have $B_2 = 0$. By $B_2 = B_3 = 0$, (36), and (38), I have $\pi_G(x_G^{**}) = \pi_B(x_B^{**}) = 0$. Then, I have $x_i^{**} = x_i^{NPV}$ $(i = G, B)$ and $B_1 = B_4 = 0$.

Next, consider the limiting case of $\lambda \to \infty$. Because of $a_G/a_B > 1$ and $\lim_{\lambda\to\infty}(r-\mu)/\lambda q = 0$, (26) is satisfied in the limiting case. Then, by Proposition 3, I have $x_G^{**} = x_G^{*}(x;\lambda q)$, $W(x) = \tilde{W_B}(x) = V_G(x;\lambda q)$, and $\tilde{W_G}(x) = \tilde{V_G}(x;\lambda q)$. The limiting results follow from Proposition 2. The proof is complete. □

Proposition 4, as well as Proposition 2, shows that the illiquidity solution fills the gaps between the NPV and ROV solutions. Indeed, as λ decreases (increases), $x_i^{**}, W(x)$, and $\tilde{W_i}(x)$ approach the NPV (ROV) solutions. When λ is sufficiently high, (26) holds, and hence the firm always forgoes a project B and invests only in a project G. This result is in line with that of the ROV model, in which the firm always invests in a project G when both projects G and B are feasible. In contrast, for a lower λ, the firm can invest in a project B if it is feasible.

Lastly, I examine the impacts of illiquidity on the investment thresholds and firm values in numerical examples. Figure 4 shows $W(x), ROV_G(x), x_G^{**}, x_B^{**}$, and x_G^{ROV} along with the liquidity cost $LC = (ROV_G(x) - W(x))/ROV_G(x)$ and the threshold decrease $TD = (x_G^{ROV} - x_G^{**})/x_G^{ROV}$ for varying levels of $1/\lambda$. The parameter values other than λ are set in Table 1. For $1/\lambda \leq 0.5$, (26) is satisfied, and hence $x_B^{**} = \infty$ holds. As shown in Proposition 4, $W(x), x_G^{**}$, and x_B^{**} monotonically decrease in $1/\lambda$. As in Figure 2, I find that the impact of illiquidity is greater on the investment thresholds than on the firm value. Indeed, I have $LC = 0.0112, TD = 0.2142$, and $x_B^{**} = 1.6409$ for $\lambda = 1$. Notably, I can see from Figure 4 that x_B^{**} decreases much more sharply than x_G^{**} for $1/\lambda \approx 0.5$ because x_B^{**}, unlike x_G^{**}, becomes infinity for $1/\lambda \leq 0.5$. In the absence of illiquidity of investment opportunities, the firm can always choose the best investment project, but with illiquidity, the firm might give up waiting for the best project and invest in the inferior project. Thus, the investment threshold for the inferior project depends greatly on the degree of illiquidity. I will verify this result in a case of continuous project types in Section 4.4.

LC and TD in Figure 4 are higher than those in Figure 2 because a project is B at probability 0.5 in this example. Note that LC does not depends on the initial value x for $x \leq x_G^{**}$ and that LC increases to $(r-\mu)/(r+\lambda q-\mu)$ for $1/\lambda \leq 0.5$ and $(r+\lambda-\mu-\lambda\bar{a}/a_G)/(r+\lambda-\mu)$ for $1/\lambda > 0.5$ as $x \to \infty$. LC can be significantly high for plausible levels of $1/\lambda$ when $r-\mu$ is quite large.[12] By Proposition 4, I can also show that $\lim_{\lambda\to 0} LC = 1, \lim_{\lambda\to 0} TD = 1/\beta$, and $\lim_{\lambda\to\infty} LC = \lim_{\lambda\to\infty} TD = 0$.

Figure 5 shows $W(x), ROV_G(x), x_G^{**}, x_B^{**}$, and x_G^{ROV} for varying levels of σ. The parameter values other than σ are set in Table 1. Note that (26) does not hold, and hence, I have finite x_B^{**}. The impacts of σ on $W(x)$ and x_G^{**} are weaker than those in the ROV model because the illiquid model is intermediate between the ROV and NPV models. Then, LC and TD in Figure 5 monotonically increase in σ. These findings are similar to those of the baseline model (cf. Figure 3). In conclusion, I argue that the effects of illiquidity of investment opportunities increase with a higher volatility of the project payoff.

Although I cannot derive any closed-form expression of $\lim_{\sigma\to 0} x_G^{**}$ for $\mu > 0$, it follows from (29) that $\lim_{\sigma\to 0} x_G^{**} = x_G^{NPV}$ for $\mu \leq 0$ and $\lim_{\sigma\to\infty} x_i^{**} = x_i^{NPV}(r+\lambda-\mu)/(r+\lambda-\mu-\lambda\bar{a}/a_i)$ $(i = G, B)$. Then, I can show that $\lim_{\sigma\to 0} TD = 0$ for $\mu < 0$, $\lim_{\sigma\to\infty} LC = (r+\lambda-\mu-\lambda\bar{a}/a_G)/(r+\lambda-\mu)$, and $\lim_{\sigma\to\infty} TD = 1$.

4 Extensions

4.1 Search cost

In this subsection, I show that the same analysis as in Sections 3.3 and 3.4 can be applied to the illiquid model with cost flows of searching for investment opportunities. In mergers and acquisitions, the acquiring firm might continue costly searches and negotiations until it succeeds. A financially constrained firm might continue to apply for government grants until it receives sufficient funds. To capture these situations, in addition to the baseline model, I assume that the firm incurs cost flows, denoted by c, when searching for a project. The firm can freely suspend the search and switch between search and suspension. I use subscript c to stand for the case with search cost c, instead of omitting project type i.

As in Section 3.3, I can show that a threshold policy is optimal, and I have two ODEs. ODE (9) does not change because the firm does not search a project for $X(t)$ in the

[12] If I set $\mu = -0.05$ in Table 1, I have $LC = 0.2143$ and $TD = 0.124$.

rejection region. ODE (10) changes to

$$\mu x V_c'(x) + 0.5\sigma^2 x^2 V_c''(x) + \lambda(\pi(x) - V_c(x)) - c = rV_c(x) \quad (x > x_c^*) \tag{40}$$

because the firm has to pay cost flows c in the acceptance region. As in the proof of Proposition 1, by solving ODEs (9) and (40) with the continuous differentiability condition at x_c^*, as well as the boundary conditions $\lim_{x\to 0} V_c(x) = 0, \lim_{x\to\infty} V_c(x)/\pi(x) < \infty, V_c(x_c^*) = \pi(x_c^*)$, I can show the following proposition.

Proposition 5

$$V_c(x) = \begin{cases} A_{1c}x^{\beta} & (x < x_c^*), \\ \dfrac{\lambda a x}{(r-\mu)(r+\lambda-\mu)} - \dfrac{\lambda I + c}{r+\lambda} + A_{2c}x^{\gamma_\lambda} & (x \geq x_c^*), \end{cases} \tag{41}$$

where the investment threshold x_c^* and coefficients A_{1c}, A_{2c} are

$$x_c^* = \frac{(r-\mu)(r+\lambda-\mu)(((\beta-\gamma_\lambda)r+\beta\lambda)I - \gamma_\lambda c)}{a(r+\lambda)((\beta-\gamma_\lambda)(r-\mu)+(\beta-1)\lambda)}, \tag{42}$$

$$A_{1c} = \frac{\pi(x_c^*)}{x_c^{*\beta}},$$

$$A_{2c} = \frac{1}{x_c^{*\gamma_\lambda}}\left(\frac{a x_c^*}{r+\lambda-\mu} - \frac{rI+c}{r+\lambda}\right).$$

I omit the proof of Proposition 5 because it can be proved in the same fashion as in the proof of Proposition 1. As in Proposition 1, the firm invests in a project only when it arrives at time t satisfying $X(t) \geq x_c^*$. Note that $-\gamma_\lambda c > 0$ of the denominator in (42). Because of cost flows c, the investment threshold x_c^* is higher than that of the baseline model. The upper equation in (41) stands for the value of waiting when the option value is higher than the investment payoff, i.e, $V_c(x) > \pi(x)$. The lower equation in (41) stands for the value of waiting when the option value is lower than the investment payoff, i.e, $V_c(x) \leq \pi(x)$. The first and second terms of the lower equation stand for the expected payoff of continuing search and investing whenever a project is found. The last term $A_{2c}x^{\gamma_\lambda}$ stands for the value of the option to suspend search when $X(t)$ passes x_c^* before the arrival of a project. I can show similar results to Propositions 2–4, although I do not redundantly present them. To summarize, the main results in Sections 3.3 and 3.4 remain unchanged with the presence of search cost.

4.2 The relationship with the regime-switching model

This subsection reveals the relationship between the baseline model in Section 3.3 and the regime-switching model of Hackbarth, Miao, and Morellec (2006). I consider the following regime-switching model with two regimes. In one of the regimes (say, boom), the firm can invest in a project at an arbitrary time, whereas in the other regime (say, bust), the firm has no chance to invest in a project. For notational simplicity, I skip the subscript i for the project type. The project payoff function $\pi(x)$ is the same as in Section 3. A bust regime switches to a boom regime following a Poisson process with the arrival rate $\lambda(>0)$, whereas a boom regime switches to a bust regime following a Poisson process with the arrival rate $\eta(>0)$. I assume that the Poisson processes are independent of the state process $X(t)$. I denote by $V_r(x)$ and $\tilde{V}_r(x)$ the firm values in bust and boom regimes, respectively. The subscript r stands for the regime-switching model. The regime-switching model can be regarded as a simplified version of Hackbarth, Miao, and Morellec (2006)'s model in which cash flows from investment depend on the regimes.

As in Section 3.3 (see Hackbarth, Miao, and Morellec (2006) and Bensoussan, Yan, and Yin (2012) for more general cases), the following ODEs hold:

$$\mu x V_r'(x) + 0.5\sigma^2 x^2 V_r''(x) + \lambda(\tilde{V}_r(x) - V_r(x)) = rV_r(x), \quad (x < x_r^*), \tag{43}$$

$$\mu x V_r'(x) + 0.5\sigma^2 x^2 V_r''(x) + \lambda(\pi(x) - V_r(x)) = rV_r(x), \quad (x > x_r^*), \tag{44}$$

$$\mu x \tilde{V}_r{}'(x) + 0.5\sigma^2 x^2 \tilde{V}_r{}''(x) + \eta(V_r(x) - \tilde{V}_r(x)) = r\tilde{V}_r(x) \quad (x < x_r^*), \tag{45}$$

where x_r^* is the investment threshold in a boom regime. The terms $\lambda(\tilde{V}_r(x) - V_r(x))$ in (43) and $\lambda(\pi(x) - V_r(x))$ in (44) mean that $V_r(x)$ changes to $\tilde{V}_r(x)$ and $\pi(x)$ when a bust regime switches to a boom regime. The term $\eta(V_r(x) - \tilde{V}_r(x))$ in (45) means that $\tilde{V}_r(x)$ changes to $V_r(x)$ when a boom regime switches to a bust regime. The boundary conditions are $\lim_{x\to 0} V_r(x) = 0, \lim_{x\to\infty} V_r(x)/\pi(x) < \infty, \lim_{x\to 0} \tilde{V}_r(x) = 0, \tilde{V}_r(x) = \pi(x) \ (x \geq x_r^*)$, and the the continuous differentiability of $V_r(x)$ and $\tilde{V}_r(x)$ at $x = x_r^*$.[13]

As in the proof of Proposition 1, by solving ODEs (43)–(45) with the conditions above, I can show the following proposition.

[13]The continuous differentiability of $\tilde{V}_r(x)$ at $x = x_r^*$ corresponds to the smooth pasting condition in Dixit and Pindyck (1994), although the continuous differentiability of $V_r(x)$ at $x = x_r^*$ follows from Theorem 4.4.9 in Karatzas and Shreve (1998). See also Hackbarth, Miao, and Morellec (2006) and Bensoussan, Yan, and Yin (2012).

Proposition 6

$$V_r(x) = \begin{cases} A_{1r}x^{\beta} - \dfrac{\lambda}{\eta}A_{2r}x^{\beta_{\lambda+\eta}} & (x < x_r^*), \\ \dfrac{\lambda a x}{(r-\mu)(r+\lambda-\mu)} - \dfrac{\lambda I}{r+\lambda} + A_{3r}x^{\gamma_\lambda} & (x \geq x_r^*), \end{cases} \tag{46}$$

$$\tilde{V}_r(x) = \begin{cases} A_{1r}x^{\beta} + A_{2r}x^{\beta_{\lambda+\eta}} & (x < x_r^*), \\ \pi(x) & (x \geq x_r^*), \end{cases} \tag{47}$$

where the investment threshold x_r^* and coefficients A_{1r}, A_{2r}, A_{3r} are

$$x_r^* = \frac{(r+\lambda-\mu)(\beta(r+\lambda)(\beta_{\lambda+\eta}\eta + \lambda(\beta_{\lambda+\eta} - \gamma_\lambda)) - \gamma_\lambda\eta(\beta_{\lambda+\eta}r + \beta\lambda))x^{NPV}}{(r+\lambda)(\lambda(\beta-1)(\beta_{\lambda+\eta} - \gamma_\lambda)(r+\lambda+\eta-\mu) + \eta(\beta-\gamma_\lambda)(\beta_{\lambda+\eta}-1)(r-\mu))},$$

$$A_{1r} = \frac{1}{(\beta_{\lambda+\eta} - \beta)x_r^{*\beta}}\left(\frac{(\beta_{\lambda+\eta}-1)ax_r^*}{r-\mu} - \beta_{\lambda+\eta}I\right),$$

$$A_{2r} = \frac{1}{(\beta_{\lambda+\eta} - \beta)x_r^{*\beta_{\lambda+\eta}}}\left(\beta I - \frac{(\beta-1)ax_r^*}{r-\mu}\right),$$

$$A_{3r} = \frac{1}{x_r^{*\gamma_\lambda}}\left(A_{1r}x_r^{*\beta} - \frac{\lambda}{\eta}A_{2r}x_r^{*\beta_{\lambda+\eta}} - \frac{\lambda a x_r^*}{(r-\mu)(r+\lambda-\mu)} + \frac{\lambda I}{r+\lambda}\right).$$

I omit the proof of Proposition 6 because the calculation is lengthy but can be done in the same manner as in the proof of Proposition 1. Notably, Proposition 6 shows the closed-form solution, although Hackbarth, Miao, and Morellec (2006) do not derive any closed-form solution. In a boom regime, the firm invests in the project at time t satisfying $X(t) \geq x_r^*$. The investment time is either the first hitting time to x_r^* or a boom arrival time t satisfying $X(t) \geq x_r^*$.

The firm value in a bust regime, $V_r(x)$ in (46), is similar to $V_i(x)$ in (11) because the firm cannot invest in the project in a bust regime. The upper equation in (46) stands for the value of waiting when the option value is higher than the investment payoff, i.e, $V_r(x) > \pi(x)$. Because of the transition probability from a bust to a boom, unlike in (11), it has an extra term $A_{2r}x^{\beta_{\lambda+\eta}}\lambda/\eta$. The lower equation in (46) stands for the value of waiting when the option value is lower than the investment payoff, i.e, $V_r(x) \leq \pi(x)$. The first and second terms of the lower equation stand for the expected payoff of investing whenever a boom regime arrives. The last term $A_{3,r}x^{\gamma_\lambda}$ stands for the value of the option to change the policy when $X(t)$ passes x_r^*.

On the other hand, the firm value in a boom regime, $\tilde{V}_r(x)$ in (47) is similar to $ROV_i(x)$ in (5) because the firm can invest in the project at an arbitrary time in a boom regime. The upper equation in (47) stands for the value of waiting when the option value is higher than the investment payoff, i.e, $V_r(x) > \pi(x)$. Because of the transition probability from a boom to a bust, unlike in (5), it has an extra term $A_{2r}x^{\beta_{\lambda+\eta}}$. The lower equation in (47) represents the investment payoff.

By virtue of the closed-form expression of x_r^*, I can directly show that $\lim_{\eta\to 0} x_r^* = x^{ROV}$ and $\lim_{\eta\to\infty} x_r^* = x^*$. Then, I can easily show the following proposition.

Proposition 7 *$V_r(x)$ and $\tilde{V}_r(x)$, and x_r^* monotonically decrease in η.*

$$\lim_{\eta\to 0} \tilde{V}_r(x) = ROV(x),\ \lim_{\eta\to 0} x_r^* = x^{ROV}$$
$$\lim_{\eta\to\infty} V_r(x) = V(x),\ \lim_{\eta\to\infty} \tilde{V}_r(x) = \tilde{V}(x),\ \lim_{\eta\to\infty} x_r^* = x^*.$$

Proposition 7 shows that the regime-switching model is intermediate between the ROV model and the baseline model in Section 3.3. Indeed, the solution of the regime-switching model converges to that of the ROV model for lower transition probability from a boom to a bust, η, and converges to that of the baseline model for a higher η. Intuitively, with a higher η, a boom ends sooner, and hence, the value of deferring investment within a boom regime decreases. In the limiting case of $\eta \to \infty$, the firm has to decide whether or not to invest as soon as a boom regime arrives, which leads to the same solution as the baseline solution in Section 3.3. Thus, the regime-switching model is regarded as an extended model of the illiquid model in Section 3.3.[14]

4.3 Put-type option

Although this paper has examined call-type real options so far, I can also derive the solutions for put-type real options, such as asset sales, liquidation, and bankruptcy options, in the same fashion as in Sections 3.3 and 3.4. This subsection examines the stylized model below. Suppose that a firm has an illiquid and indivisible asset that generates continuous streams of cash flows $aX(t)$, where a is a positive constant. The firm occasionally meets acquirers who want to buy the asset at a constant price $I(> 0)$ and optimizes the asset sales policy.[15] Assume that the acquirers' arrival process is a Poisson process with the arrival rate λ. If the firm rejects an acquirer's offer, the firm has to wait until another acquirer arrives. The Poisson arrival process stands for illiquidity of asset sales opportunities.

[14] Similarly, the regime-switching model with three regimes can be regarded as an extension of the illiquid model with two project types in Section 3.4.

[15] I can similarly derive the closed-form solution when the sales price is expressed as a linear function of $X(t)$.

As in Section 3.3, I can show that a threshold policy is optimal and that the firm's asset value $V_p(x)$ satisfies

$$\mu x V_p'(x) + 0.5\sigma^2 x^2 V_p''(x) + \lambda(I - V_p(x)) + ax = rV(x) \quad (x < x_p^*), \tag{48}$$

$$\mu x V_p'(x) + 0.5\sigma^2 x^2 V_p''(x) + ax = rV(x) \quad (x > x_p^*), \tag{49}$$

where x_p^* denotes the asset sales threshold below which the firm wishes to sell the asset. The subscript p stands for the put-type option. Note that ax in (48) and (49) corresponds to cash flows from the asset, while $\lambda(I - V_p(x))$ corresponds to the fact that $V_p(x)$ changes to I with probability $\lambda \mathrm{d}t$ in infinitesimal time interval $\mathrm{d}t$.

As in the proof of Proposition 1, by solving ODEs (48) and (49) with the continuous differentiability condition at x_p^* along with the boundary conditions $\lim_{x\to 0} V_p(x) < \infty, \lim_{x\to\infty} V_p(x)/ax < \infty, V_p(x^*) = I$, I can show the following proposition.

Proposition 8

$$V_p(x) = \begin{cases} \dfrac{ax}{r+\lambda-\mu} + \dfrac{\lambda I}{r+\lambda} + A_{2p}x^{\beta_\lambda} & (x \le x_p^*), \\ \dfrac{ax}{r-\mu} + A_{1p}x^{\gamma} & (x > x_p^*), \end{cases} \tag{50}$$

where the asset sales threshold x^* and coefficients A_{1p}, A_{2p} are

$$\begin{aligned} x_p^* &= \frac{(r-\mu)(r+\lambda-\mu)((\gamma-\beta_\lambda)r + \gamma\lambda)I}{a(r+\lambda)((\gamma-\beta_\lambda)(r-\mu) + (\gamma-1)\lambda)}, \\ A_{1p} &= \frac{1}{x_p^{*\gamma}}\left(I - \frac{ax_p^*}{r-\mu}\right), \\ A_{2p} &= \frac{1}{x_p^{*\beta_\lambda}}\left(\frac{rI}{r+\lambda} - \frac{ax_p^*}{r+\lambda-\mu}\right). \end{aligned}$$

The proof of Proposition 8 can be done in the same manner as in the proof of Proposition 1. The firm liquidates the asset only when an acquirer arrives at time t satisfying $X(t) \le x_p^*$. I can easily show that the asset sales threshold is higher than that of the liquid model. In other words, lower asset liquidity decreases the option value of waiting, and hence, the firm more eagerly accepts an acquirer's offer.

The lower equation represents the value of waiting when the firm's asset value is higher than the sales proceeds, i.e, $V_p(x) > I$. More precisely, the first term represents the asset value without the asset sales opportunity, and the second term represents the value of the asset sales option. The upper equation in (50) stands for the value of waiting when the asset value is lower than the sales proceeds, i.e, $V_p(x) \le I$. The first and second terms of the upper equation stand for the expected payoff when the firm sells the asset whenever an acquirer arrives, whereas the last term $A_{2p}x^{\beta_\lambda}$ stands for the value of the option to suspend the asset sales at time t satisfying $X(t) > x_p^*$. I can show results analogous to Proposition 2-7, although I do not redundantly present them. Thus, for put-type real options, the illiquid model also fills the gaps in the NPV, ROV, and regime-switching models.

4.4 A continuum of project types

This subsection extends the illiquid model with two project types in Section 3.4 to a case with a continuum of project types. Suppose that projects with payoff $\pi_k(x) = a_k x/(r-\mu) - I_k$ ($k \in [0,1]$) may arrive, where I define $a_k = ka_G + (1-k)a_B$ and $I_k = kI_G + (1-k)I_B$. Then, $\pi_k(x)$ increases in k from the the worst type $k = 0$ (i.e., project B in Section 3.4) to the best type $k = 1$ (i.e., project G). For simplicity, I assume that project type k follows a uniform distribution on $[0,1]$, independently of $X(t)$.

Although I cannot analytically derive the solution, I can numerically compute the solution to ODE

$$\mu x U'(x) + 0.5\sigma^2 x^2 U''(x) + \lambda(1-k^*(x))(\pi_{0.5+0.5k^*(x)}(x) - U(x)) = rU(x), \qquad (51)$$

where $U(x)$ denotes the firm value when no project is feasible, and I define $k^*(x) = \min\{k \in [0,1] \mid U(x) \le \pi_k(x)\}$. If $U(x) > \pi_1(x)$ holds, I artificially define $k^*(x) = 1$ to remove the term $\lambda(1-k^*(x))$. The type $k^*(x)$ stands for the lowest acceptable type for x. In other words, the firm undertakes only a project of type $k \ge k^*(X(t))$ and forgoes a project of type $k < k^*(X(t))$ when it arrives at time t. From the opposite perspective, the investment threshold for type k becomes the inverse function $k^{*-1}(k)$. In (51), $\lambda(1-k^*(x))(\pi_{0.5+0.5k^*(x)}(x) - U(x))$ corresponds to the fact that the firm value $U(x)$ changes to $\pi_{0.5+0.5k^*(x)}(x)$ with probability $\lambda(1-k^*(x))\mathrm{d}t$ in infinitesimal time interval $\mathrm{d}t$. Note that $\pi_{0.5+0.5k^*(x)}(x)$ and $\lambda(1-k^*(x))\mathrm{d}t$ represent the mean payoff and the arrival probability of the acceptable types, respectively.

Figure 6 shows the lowest acceptable type $k^*(x)$ for $1/\lambda = 0.6, 0.8$, and 1. I can see from the figure that lower liquidity decreases $k^*(x)$. Lower liquidity decreases the value of waiting for a superior project, and the firm accepts a project more eagerly. This result is consistent with Proposition 3 (cf. (26)). In the figure, I also find that the differences of the three lines are larger for lower $k^*(x)$. This means that the impact of illiquidity on the investment threshold for type k, i.e., $k^{*-1}(k)$, decreases in k. This result shows the robustness of Section 3.4's result that the investment threshold for an inferior project is more affected by the degree of illiquidity. Because the results of LC and TD and the comparative statics results of σ are similar to those of Sections 3.3 and 3.4, I omit the figures.

5 Conclusion

This paper investigated the effects of illiquidity of investment opportunities on investment decisions. I derived the firm value and investment threshold in the closed forms when investment opportunities follow a Poisson process. Lower liquidity decreases the option value of waiting and the hurdle rate for investment. For cash flows with a positive growth rate, the impact of illiquidity on the investment threshold is greater than on the firm value. A higher cash flow volatility amplifies the impacts of illiquidity on the firm value and investment threshold.

Furthermore, I extended the results to a case with two types of projects arriving randomly. Although I could not derive a closed-form solution, I derived the analytical solution. When the profitability gap between the two types, liquidity, and the probability of the good-type appearance are higher, the firm is more likely to accept only a good-type project and forgo a bad-type project. When these factors are low, the firm accepts only a good-type project for intermediate levels of the state variable and accepts even a bad-type project for sufficiently high levels of the state variable. The impact of illiquidity on the bad-type investment threshold is much greater than on the good-type investment threshold.

I also proved that the solution of the illiquid model converges to those of the NPV and ROV models for lower and higher liquidity, respectively. I also showed that the solution agrees with the limit of the corresponding regime-switching model. Thus, this paper fills the gaps in the NPV, ROV, and regime-switching models in terms of illiquidity of option exercise opportunities. The proposed models and solutions can be a new framework to study real options with illiquidity of option exercise opportunities. Potential applications include but are not limited to mergers and acquisitions with illiquidity of searching and matching, illiquid asset sales, and corporate investment with illiquidity of fundraising.

A The continuity of $\tilde{V}_i(x)$ and the optimality of a threshold policy

First, we will show

$$0 \leq \tilde{V}_i(x+\Delta) - \tilde{V}_i(x) \leq \frac{\Delta a_i}{r-\mu} \quad (x, \Delta \in \mathbb{R}_+) \tag{52}$$

as follows. We denote the set of $\mathcal{F}_t$-stopping times by $\mathcal{T}$. We also denote by $\mathcal{T}_\lambda^P$ the subset of $\mathcal{T}$ restricted to the Poisson arrival times with the arrival rate λ. Assume that the Poisson process arrives at time $t = 0$. For $x, \Delta \in \mathbb{R}_+$, I have

$$\begin{aligned}
&\tilde{V}_i(x+\Delta) \\
&= \sup_{\tau \in \mathcal{T}_\lambda^P} \mathbb{E}[\mathrm{e}^{-r\tau} \left(\frac{a_i(x+\Delta)\mathrm{e}^{(\mu-0.5\sigma^2)\tau+\sigma B(\tau)}}{r-\mu} - I \right)] \\
&\leq \sup_{\tau \in \mathcal{T}_\lambda^P} \mathbb{E}[\mathrm{e}^{-r\tau} \left(\frac{a_i x\mathrm{e}^{(\mu-0.5\sigma^2)\tau+\sigma B(\tau)}}{r-\mu} - I \right)] + \sup_{\tau \in \mathcal{T}_\lambda^P} \mathbb{E}[\mathrm{e}^{-r\tau} \left(\frac{a_i \Delta\mathrm{e}^{(\mu-0.5\sigma^2)\tau+\sigma B(\tau)}}{r-\mu} \right)] \\
&\leq \sup_{\tau \in \mathcal{T}_\lambda^P} \mathbb{E}[\mathrm{e}^{-r\tau} \left(\frac{a_i x\mathrm{e}^{(\mu-0.5\sigma^2)\tau+\sigma B(\tau)}}{r-\mu} - I \right)] + \sup_{\tau \in \mathcal{T}} \mathbb{E}[\mathrm{e}^{-r\tau} \left(\frac{a_i \Delta\mathrm{e}^{(\mu-0.5\sigma^2)\tau+\sigma B(\tau)}}{r-\mu} \right)] \\
&= \tilde{V}_i(x) + \frac{a_i\Delta}{r-\mu}.
\end{aligned}$$

Similarly, I can easily show that $\tilde{V}_i(x+\Delta) \geq \tilde{V}_i(x)$ by $\mathrm{e}^{-r\tau} a_i \Delta \mathrm{e}^{(\mu-0.5\sigma^2)\tau+\sigma B(\tau)}/(r-\mu) \geq 0$. Then, I have (52).

Note that (52) shows the continuity of the value function $\tilde{V}_i(x)$. By the continuity of $\tilde{V}_i(x)$ and $\pi_i(x)$, $S_i^* = \{x \in \mathbb{R}_+ \mid \tilde{V}_i(x) = \pi_i(x)\}$ is a closed set. Because $\tilde{V}_i(x) \leq ROV_i(x) = \pi_i(x)$ holds for $x \geq x_i^{ROV}$, S_i^* includes $\{x \in \mathbb{R}_+ \mid x \geq x_i^{ROV}\}$. Because of $\tilde{V}_i(x) \geq 0$, S_i^* is included by $\{x \in \mathbb{R}_+ \mid x \geq x_i^{NPV}\}$. Therefore, by the closeness of S_i^*, $\min\{x \in S_i^*\}$ exists between x_i^{NPV} and x_i^{ROV}. I denote this minimum by x_i^*. I can show that the optimal policy is the threshold policy, i.e., $S_i^* = \{x \in \mathbb{R}_+ \mid x \geq x_i^*\}$ as follows. Indeed, by (52), for $x > x_i^*$, I have

$$\begin{aligned}
\tilde{V}_i(x) &\leq \tilde{V}_i(x_i^*) + \frac{(x_i^* - x)a_i}{r-\mu} \\
&= \pi_i(x_i^*) + \frac{(x_i^* - x)a_i}{r-\mu} \\
&= \pi_i(x),
\end{aligned}$$

which implies $x \in S_i^*$. The proof is complete. □

Although this paper presented the simple proof in the standard manner (e.g., see Detemple (2006)), Perez and Yamazaki (2018) proved the properties in a more general case in a different manner.

B The continuity of $\tilde{W}_i(x)$ and the optimality of a threshold policy

In the same fashion as in Appendix A, we can show

$$0 \le \tilde{W}_G(x+\Delta) - \tilde{W}_G(x) \le \frac{\Delta a_G}{r-\mu} \quad (x, \Delta \in \mathbb{R}_+) \tag{53}$$

and by (53), I can show the continuity of $\tilde{W}_i(x)$, the optimality of a threshold policy, and the existence of the optimal threshold $x_G^{**} \in [x_G^{NPV}, x_G^{ROV}]$.

Next, suppose that (26) does not hold. Then, I have

$$a_B - \frac{a_G \lambda q}{r + \lambda q - \mu} > 0. \tag{54}$$

I will show

$$0 \le \tilde{W}_B(x+\Delta) - \tilde{W}_B(x) \le \frac{\Delta a_B}{r-\mu} \quad (x, \Delta \in \mathbb{R}_+). \tag{55}$$

as follows. Below, I assume that the Poisson process arrives at time $t = 0$ with $i(0) = B$, where $i(\tau)$ denotes the feasible project type at the Poisson arrival time τ.

$$\begin{aligned}
&\tilde{W}_B(x+\Delta) \\
=& \sup_{\tau \in \mathcal{T}_\lambda^P} \mathbb{E}[\mathrm{e}^{-r\tau} \left(\frac{a_{i(\tau)}(x+\Delta)\mathrm{e}^{(\mu-0.5\sigma^2)\tau+\sigma B(\tau)}}{r-\mu} - I \right)] \\
\le& \sup_{\tau \in \mathcal{T}_\lambda^P} \mathbb{E}[\mathrm{e}^{-r\tau} \left(\frac{a_{i(\tau)}x\mathrm{e}^{(\mu-0.5\sigma^2)\tau+\sigma B(\tau)}}{r-\mu} - I \right)] + \sup_{\tau \in \mathcal{T}_\lambda^P} \mathbb{E}[\mathrm{e}^{-r\tau} \left(\frac{a_{i(\tau)}\Delta\mathrm{e}^{(\mu-0.5\sigma^2)\tau+\sigma B(\tau)}}{r-\mu} \right)] \\
\le& \tilde{W}_B(x) + \sup_{\tau \in \mathcal{T}} \mathbb{E}[1_{\{\tau<T\}}\mathrm{e}^{-r\tau} \left(\frac{a_B\Delta\mathrm{e}^{(\mu-0.5\sigma^2)\tau+\sigma B(\tau)}}{r-\mu} \right) + 1_{\{\tau \ge T\}}\mathrm{e}^{-rT} \left(\frac{a_G\Delta\mathrm{e}^{(\mu-0.5\sigma^2)T+\sigma B(T)}}{r-\mu} \right)] \qquad (56) \\
=& \tilde{W}_B(x) + \sup_{\tau \in \mathcal{T}} \mathbb{E}[\mathrm{e}^{-(r+\lambda q)\tau} \left(a_B - \frac{a_G\lambda q}{r+\lambda q-\mu} \right) \left(\frac{\Delta\mathrm{e}^{(\mu-0.5\sigma^2)\tau+\sigma B(\tau)}}{r-\mu} \right)] + \frac{a_G\lambda q\Delta}{(r-\mu)(r+\lambda q-\mu)} \qquad (57) \\
=& \tilde{W}_B(x) + \left(a_B - \frac{a_G\lambda q}{r+\lambda q-\mu} \right) \frac{a_B\Delta}{r-\mu} + \frac{a_G\lambda q\Delta}{(r-\mu)(r+\lambda q-\mu)} \\
=& \tilde{W}_B(x) + \frac{\Delta a_B}{r-\mu},
\end{aligned}$$

where in (56), T denotes the first arrival time of the Poisson process with the arrival rate λq (i.e., project G). The optimal stopping problem in (56) is the problem in which the firm can gain either the bad-type payoff $a_B\Delta\mathrm{e}^{(\mu-0.5\sigma^2)\tau+\sigma B(\tau)}/(r-\mu)$ at the optimal time τ or the good-type payoff $a_G\Delta\mathrm{e}^{(\mu-0.5\sigma^2)T+\sigma B(T)}/(r-\mu)$ at the first arrival time of project G. Note that the firm has no incentive to defer the good-type payoff

$a_G \Delta \mathrm{e}^{(\mu - 0.5\sigma^2)T + \sigma B(T)}/(r-\mu)$. In (57), the second term represents the option value of gaining the bad-type payoff before the first arrival time of project G, whereas the last term represents the value of the good-type payoff at the first arrival time. Because of (54), the optimal stopping time in (57) is equal to 0, which implies that the firm has no incentive to defer the bad-type payoff. I can easily show that $\tilde{W}_B(x+\Delta) \geq \tilde{W}_B(x)$ by $\mathrm{e}^{-r\tau} a_{i(\tau)} \Delta \mathrm{e}^{(\mu-0.5\sigma^2)\tau + \sigma B(\tau)}/(r-\mu) \geq 0$. Hence, I have (55).

By (55), I can show the continuity of $\tilde{W}_B(x)$, the optimality of a threshold policy, and the existence of the optimal threshold $x_B^{**} \geq \max(x_G^{**}, x_B^{NPV})$ in the same fashion as in Appendix A. The proof is complete. □

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Table 1: Baseline parameter values.

r	μ	σ	λ	x	a_G	I_G	a_B	I_B	q
0.07	0.05	0.2	1	0.5	1	10	1/1.02	10.2	0.5

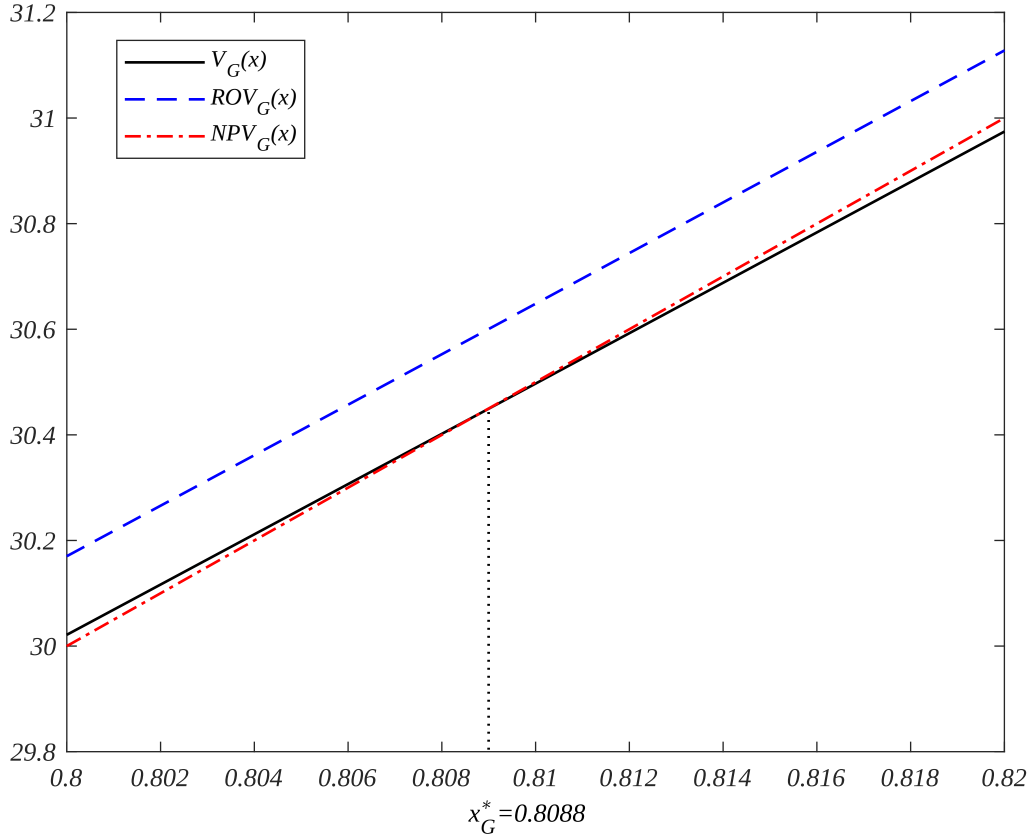

Figure 1: Firm value functions. The figure plots functions $V_G(x), ROV_G(x)$, and $NPV_G(x)$. The parameter values other than x are set in Table 1.

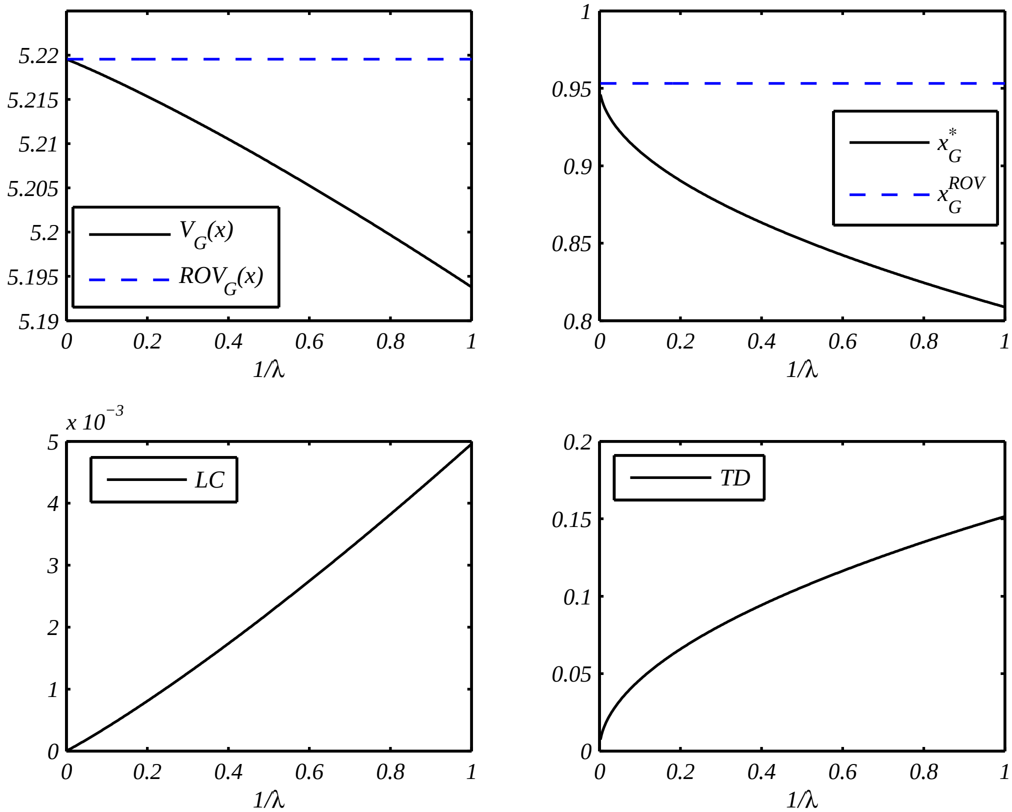

Figure 2: Comparative statics with respect to illiquidity $1/\lambda$. The figure plots $V_G(x), ROV_G(x), x_G^*, x_G^{ROV}, LC$, and TD. The other parameter values are set in Table 1.

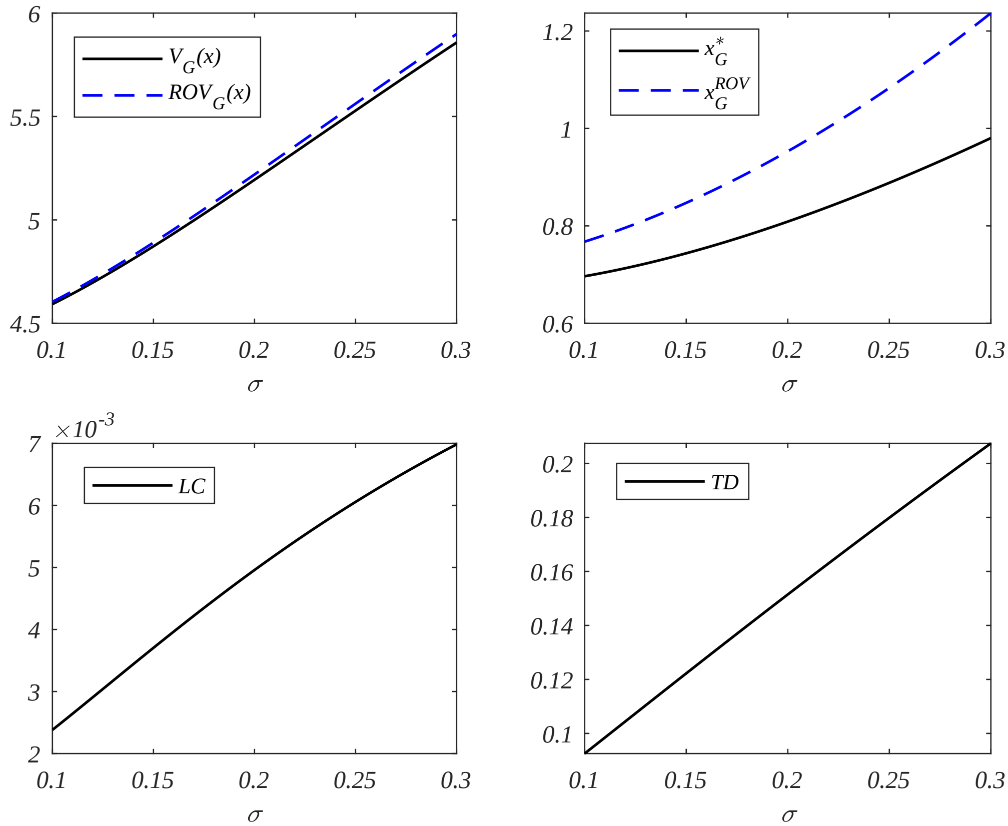

Figure 3: Comparative statics with respect to volatility σ. The figure plots $V_G(x), ROV_G(x), x_G^*, x_G^{ROV}$, LC, and TD. The other parameter values are set in Table 1.

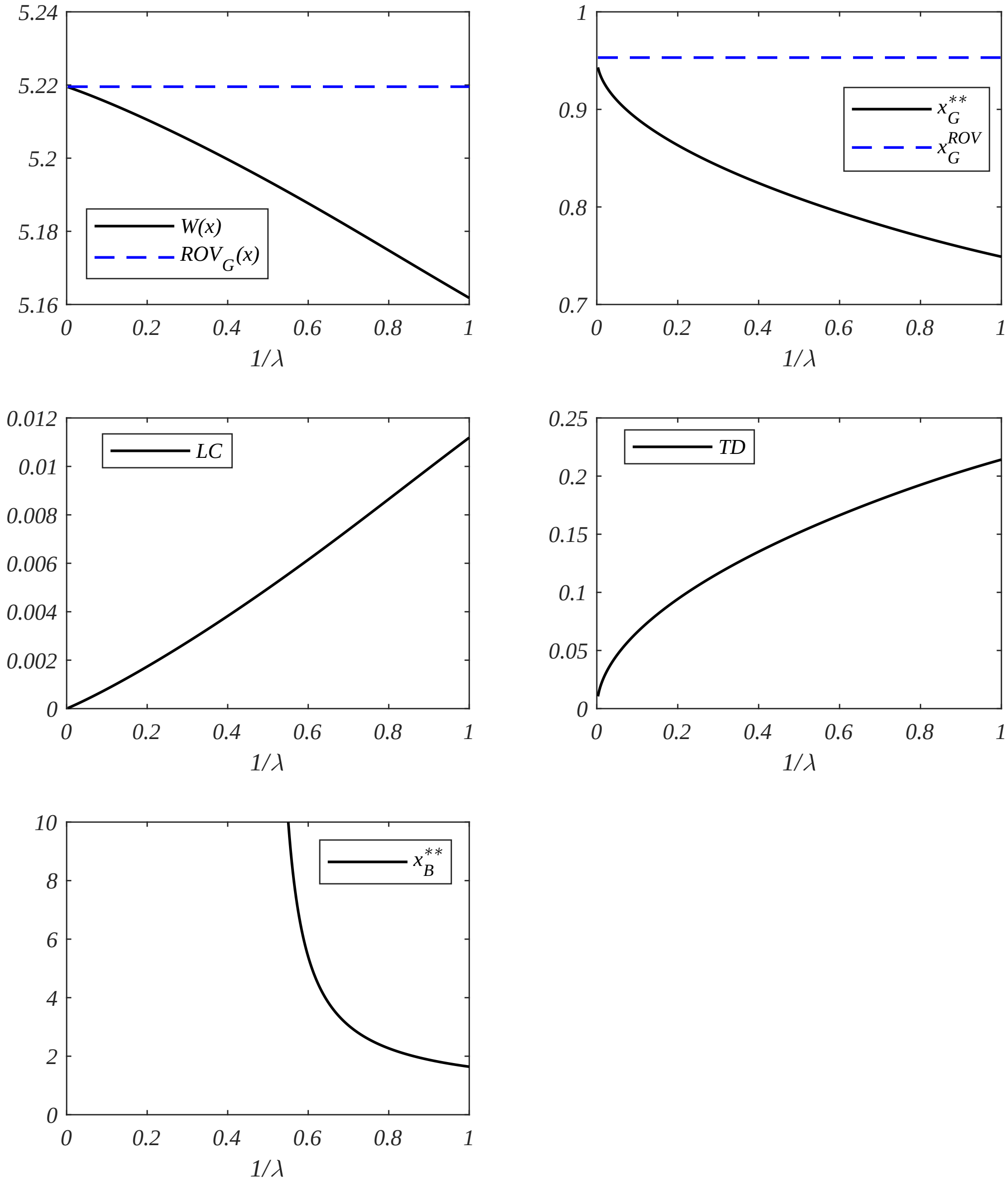

Figure 4: Comparative statics with respect to λ in the case of two project types. The figure plots $W(x), ROV_G(x), x_G^{**}, x_B^{**}, x_G^{ROV}, LC$, and TD. The parameter values are set in Table 1.

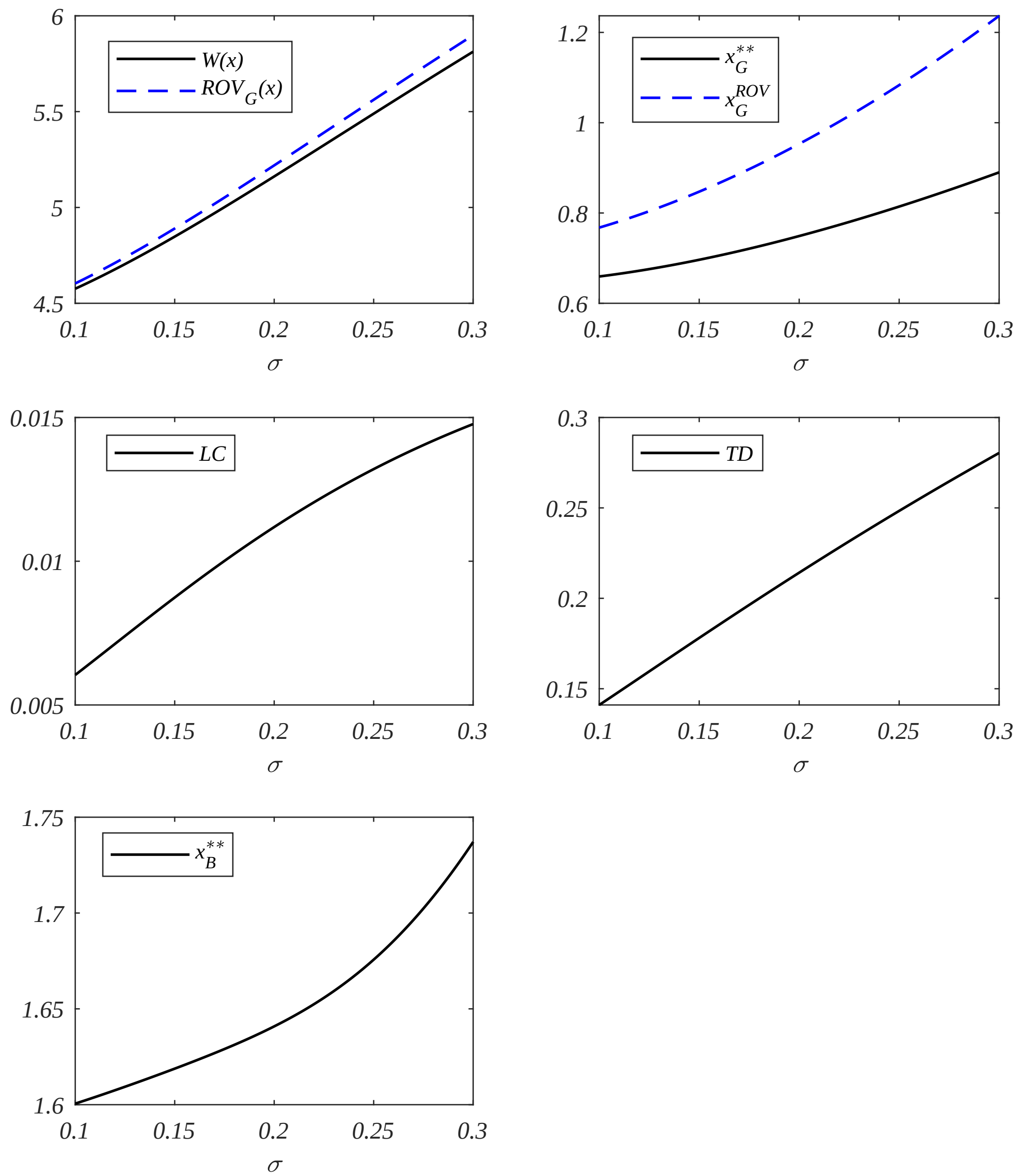

Figure 5: Comparative statics with respect to σ in the illiquid model with two project types. The figure plots $W(x), ROV_G(x), x_G^{**}, x_B^{**}, x_G^{ROV}, LC$, and TD. The other parameter values are set in Table 1.

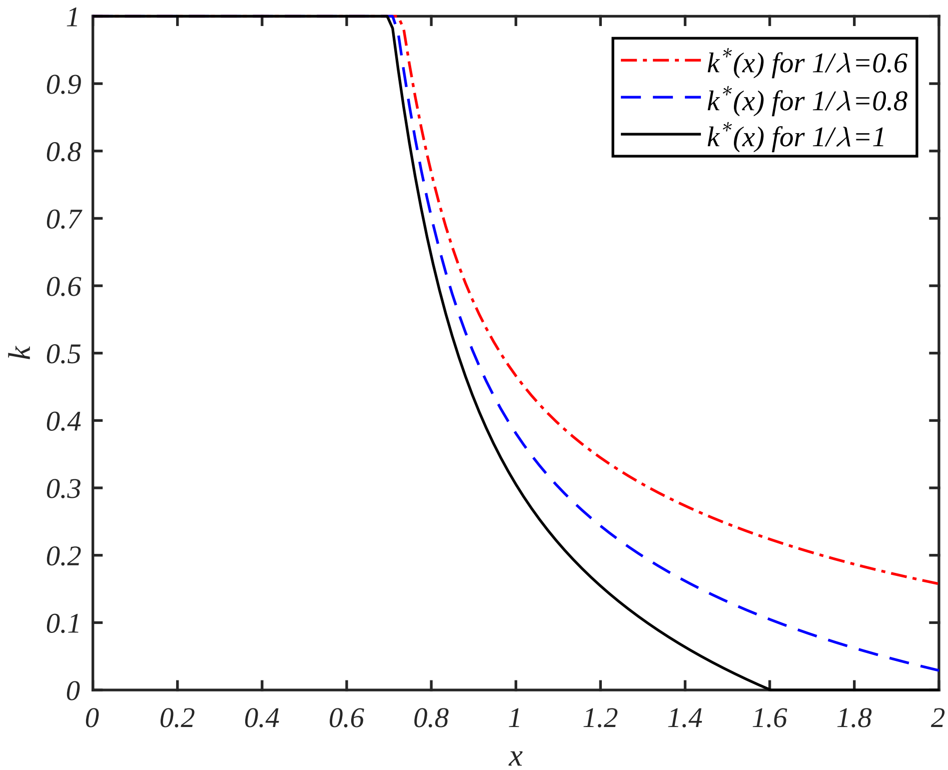

Figure 6: The lowest acceptable type $k^*(x)$ in the illiquid model with a continuum of project types. The figure plots $k^*(x)$ with respect to illiquidity $1/\lambda$. The other parameter values are set in Table 1.