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Inequality aversion with general payoff function

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Abstract

In the existing axiomatic models of inequity aversion, players have linear payoff functions, so they can predict that a dictator chooses only a completely selfish or completely fair offer in dictator games. However, experimental literature documents that a significantly amount of dictators offers 20-30% of the total pie to the passive opponent. This note, in contrast, axiomatizes inequity averse representation with general payoff function, so that we can explain such interior choices.

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Keyword: Inequality aversion under risk, Maxmin expected utility, Social preferences

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1 Introduction

Experimental literature documents that a significantly amount of subjects offers 20-30% of the total pie to passive opponents in dictator games (Camerer, 2002). This can be explained by inequilty aversion (Fehr and Schmidt, 1999) with concave payoff functions. Nevertheless, existing axiomatic models of inequity aversion limit payoff functions to be linear and cannot describe the established experimental result.

In general, an inequity averse decision maker (denoted by 1, henceforth) chooses an allocation $\mathbf{x} \in \mathbb{R}^n$ over n individuals, so that she maximizes

$$U_{FS}(\mathbf{x}) = u(x_1) - \sum_{i=2}^{n} \alpha_i \max\{u(x_i) - u(x_1), 0\} - \sum_{i=2}^{n} \beta_i \max\{u(x_1) - u(x_i), 0\} \qquad (1)$$

where $u : \mathbb{R} \to \mathbb{R}$ denotes each individual's *payoff function*, and $\alpha_i, \beta_i \in [0, 1]$ denote parameters of her envy and guilt for each i, respectively. Observe that the decision maker makes an interior offer $x_2^* \in (0, 1/2)$ in a dictator game where $n = 2$ and $x_1 + x_2 = 1$ only if u is concave. Rohde (2010) first axiomatizes representation (1) with linear payoff functions (i.e., $u(x) = x$), so her model can describe only the extreme cases $x_2^* = 0$ or $x_2^* = \frac{1}{2}$. Motivated by experimental studies of probablistic dictaor games (e.g., Brock et al. 2013), Saito (2013) considers a preference over risky allocations and axiomatizes a function combining $V(\mathbf{p}) = U_{FS}(\mathbb{E}_{\mathbf{p}}[\mathbf{x}])$ and $V(\mathbf{p}) = \mathbb{E}_{\mathbf{p}}[U_{FS}(\mathbf{x})]$. His model successfully distinguishes equality of opportunity and equality of outcome. Also in his axiomatic model, however, payoff functions are limited to be linear, and thus, his model can describe only the extremely selfish or extremely fair offer in probabilistic dictator games. Moreover, such a model with linear payoff functions is inconsist with a fact that the decision makers are often risk averse.

This note considers a preference over risky allocations as assumed by Saito (2013), and axiomatizes representation (1) with a general payoff function u. We call it an *Expected Equality Representation* (EER). Our result permits a decision maker to offer an interior donation between 0% and 50% in dictator games as experimental studies suggest. Moreover, it is consistent with the fact that the decision maker is risk averse.

Theoretically, in contrast to Rohde and Saito, who employ strong versions of *Comonotonic Independence* axiom (Schmeidler, 1989), we employ *Certainty-Independent* axiom of Gilboa and Schmeidler (1989, GS). Because the Certainty-Independent is mathematically simpler as GS claim, it permits us to analyze the general functional form. To characterize the EER, we apply Anscombe and Aumann's (1963, AA) framework of which domain is a set of acts (i.e., functions from objective states to lotteries). Because a domain of our analysis is a set of risky allocations, a set of players corresponds to that of states in AA. GS weaken AA's independence axiom and obtain kinked indifference curves. Because indifference curves of EER are also kinked in a similar way, we can obtain EER by applying GS's method.

The rest of the paper is organized as follows. Section 2 sets up the model. Section 3 analyzes it. Section 4 concludes.

2 Model

Let $I = \{1, \cdots, n\}$ be a set of individuals and Z be a set of prizes. Each $\mathbf{x} = (x_1, \cdots, x_n) \in Z^n$ is an *allocation* of payoffs among the individuals. A probability distribution p_i on Z with finite supports is called a *lottery*. The set of all lotteries is denoted by ΔZ, and write $D \coloneqq (\Delta Z)^n$. The primitive of our model is a binary relation $\succsim$ over D that describes a preference of the decision maker (denoted by $1 \in I$). As usual, we reduce $\succ$ and $\sim$ from $\succsim$, define a probability mixture $\oplus$ over D, and say that a function $f : \Delta Z \to \mathbb{R}$ is *linear* if $f(\lambda x \oplus (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y)$ for each $x, y \in D$ and $\lambda \in [0,1]$.

Definition 1. $\succsim$ has an *Expected Equality Representation* (EER) if there exist a function $u : Z \to \mathbb{R}$ and $(\alpha_i, \beta_i) \in \mathbb{R}^2$ for each $i \in I \backslash \{1\}$ such that $\sum_{i=2}^n \beta_i \leq 1$ and $\succsim$ is represented by $V(\mathbf{p}) \coloneqq U_{FS}(\mathbb{E}_{\mathbf{p}}(\mathbf{x}))$.

We impose on $\succsim$ four axioms. The first one is rationality:

Axiom 1. $\succsim$ is weak order, continuious, and monotone in a sense that

$$\forall \mathbf{p}, \mathbf{q} \in D;\ \mathbf{e}(p_1) \succsim \mathbf{e}(q_1) \Leftrightarrow (p_1, \mathbf{p}_{-1}) \succsim (q_1, \mathbf{p}_{-1}). \tag{2}$$

Note that an inequity averse individual often prefers $(0, 10) \succ (1, 9)$ in ultimatum games with $n = 2$. This violates Pareto efficiency:

$$\forall \mathbf{p}, \mathbf{q} \in D; [\forall i \in I; \mathbf{e}(p_i) \succsim \mathbf{e}(q_i)] \Rightarrow \mathbf{p} \succsim \mathbf{q}. \tag{3}$$

Therefore we impose (2) weaker than (3).

Although AA impose von Neumann-Morgenstern independence axiom, an other-regarding individual may prefer randomization such as $\frac{1}{2}(1,0) \oplus \frac{1}{2}(2,4) \succ (1,0) \sim (2,4)$. This violates the independence axiom as Diamond (1967) suggests this example in social choice literature. Hence we weaken it in GS's fassion as the following Axioms 2 and 3. To do so, let $\mathbf{e}(p) \in D$ denote an equal allocation in which each individual receives the same lottery $p \in \Delta Z$. Each $\mathbf{e}(p)$ corresponds to a "constant act" in GS.

Axiom 2. $\forall \mathbf{p}, \mathbf{q} \in D, \forall r \in \Delta Z; \mathbf{p} \sim \mathbf{q} \Rightarrow \frac{1}{2}\mathbf{p} \oplus \frac{1}{2}\mathbf{e}(r) \sim \frac{1}{2}\mathbf{q} \oplus \frac{1}{2}\mathbf{e}(r)$.

Axiom 3. $\forall \mathbf{p}, \mathbf{q} \in D; \mathbf{p} \sim \mathbf{q} \Rightarrow \frac{1}{2}\mathbf{p} \oplus \frac{1}{2}\mathbf{q} \succsim \mathbf{p}$.

Fehr-Schmidt model compares only between the decision maker and each individual. Axiom 4 implies that no relation between two others affects her ranking over allocations. Hence, it always holds if $n = 2$.

Axiom 4. $\forall \mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s} \in D$ with $\mathbf{e}(p_1) \sim \mathbf{e}(q_1)$, and $\forall i \neq 1$, $(\mathbf{p}_{-i}, r_i) \succsim (\mathbf{q}_{-i}, r_i)$ if $(\mathbf{p}_{-i}, s_i) \succsim (\mathbf{q}_{-i}, s_i)$.

3 Representation

Under the four axioms, our main result holds as follows:

Theorem 1. $\succsim$ satisfies Axioms 1-4 iff it has a EER.

Proof. First, we claim that $\succsim$ is represented by some *C-independent* function $I : D \to \mathbb{R}$ in a sense that

$$I(a\mathbf{v} + b\mathbf{1}) = aI(\mathbf{v}) + b. \tag{4}$$

By Axiom 1, there exists a continuous function $v : \{\mathbf{e}(p) \in D \mid p \in \Delta Z\} \to \mathbb{R}$ such that

$$v(p) \geq v(q) \Leftrightarrow \mathbf{e}(p) \succsim \mathbf{e}(q). \tag{5}$$

for each $p, q \in \Delta Z$. Moreover, the v can be linear by Axiom 2. By Axioms 5 and (2), there exists a function $f : (v(\Delta Z))^n \to \mathbb{R}$ such that

$$f(v(p_1), \cdots, v(p_n)) \geq f(v(q_1), \cdots, v(q_n)) \Leftrightarrow \mathbf{p} \succsim \mathbf{q} \tag{6}$$

for each $\mathbf{p}, \mathbf{q} \in D$. Given the f, define a function $I : \mathbb{R}^n \to \mathbb{R}$ such that

$$I(\lambda v(p_1), \cdots, \lambda v(p_n)) = \lambda f(v(q_1), \cdots, v(q_n))$$

for each $\lambda \in \mathbb{R}$. As shown in (iv) of GS's Lemma 3.3, our Axioms 2 implies C-independence of I.

Second, we show that $\succsim$ is represented by

$$U(\mathbf{p}) = \min_{\mathbf{k} \in \mathcal{K}} \sum_{i=1}^{n} k_i v(p_i) \tag{7}$$

for some unique, nonempty, closed, convex set $\mathcal{K} \subsetneq \mathbb{R}^n$. Without loss of generality, we consider a set given by

$$U := \{\mathbf{v} \in \mathbb{R}^n \mid f(\mathbf{v}) \geq 0\},$$

which is a closed convex cone whose vartix is $\mathbf{0}$. Therefore, we can apply supporting hyperplane theorem: there exists $\mathbf{k} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that for all $\mathbf{u} \in U$,

$$\mathbf{k} \cdot \mathbf{0} \leq \mathbf{k} \cdot \mathbf{u}. \tag{8}$$

Let $\mathcal{K}$ denote a set of all vectors $\mathbf{k}$ satisfying (8). Note that the above result of hyperplane theorem implies that $\mathcal{K} \neq \emptyset$. For $\mathbf{k} \in \mathcal{K}$, on the one hand, we have $\mathbf{k} \cdot \mathbf{u} \geq 0$. On the other hand, if $\min_{\mathbf{k} \in \mathcal{K}} \mathbf{k} \cdot \mathbf{v} < 0$, then $u \notin U$, that is, $f(u) < 0$. Therefore now we have

$$f(\mathbf{u}) = 0 \Longleftrightarrow \min_{\mathbf{k} \in \mathcal{K}} \mathbf{k} \cdot \mathbf{u} = 0. \tag{9}$$

Since we can take $\mathbf{u}' \in U$ such that $f(\mathbf{u}') = 0$ and $\mathbf{u} = \mathbf{u}' + c\mathbf{1}$ for given $\mathbf{u} \in \mathbb{R}^n$ satisfying that $f(\mathbf{u}) = c$, we have

$$\begin{aligned} f(\mathbf{u}) &= f(\mathbf{u}') + c \qquad (10) \\ &= \min_{\mathbf{k} \in \mathcal{K}} \mathbf{k} \cdot \mathbf{u}' + c \qquad (11) \\ &= \min_{\mathbf{k} \in \mathcal{K}} \mathbf{k} \cdot \mathbf{u}, \qquad (12) \end{aligned}$$

where (10) is due to (4), and (11) is derived by (9). Combining (5) and (12) yields (7). Similarly to GS, $\mathcal{K}$ is unique, nonempty, closed and convex. For each $\mathbf{k} \in \mathcal{K}$, we assume $\sum_{i=1}^{n} k_i = 1$ without loss of generality.

Third, we show that (7) is rewritten by EER. Let $\mathbf{v}, \mathbf{v}' \in \mathbb{R}^n$ satisfy that

$$(v_1 - v_i)(v_1' - v_i') > 0 \tag{13}$$

for each $i \in I$. Then, there exists $\mathbf{k} \in \mathcal{K}$ minimizing $\mathbf{k} \cdot \mathbf{v}$ and $\mathbf{k} \cdot \mathbf{v}'$ both by Axiom 4 and the properties of $\mathcal{K}$. Therefore,

$$U(\mathbf{p}) = \left(1 - \sum_{i=2}^{n} k_i^*\right) v(p_1) + \sum_{i=2}^{n} k_i^* v(p_i), \tag{14}$$

where $\mathbf{k}^* \in \arg\min_{k_i \in \mathcal{K}} \sum_{i=1}^{n} k_i v(p_i)$. Next we show that each k_i^* in $\mathbf{k}^*$ depends only on (p_1, p_i), and

$$k_i^*(p_1, p_i) = \begin{cases} \overline{k}_i & \text{if } v(p_1) \geq v(p_i) \\ \underline{k}_i & \text{if } v(p_1) \leq v(p_i) \end{cases}, \tag{15}$$

where $\underline{k}_i \leq k_i \leq \overline{k}_i$ for each $\mathbf{k} \in \mathcal{K}$ and each $i \in I \backslash \{1\}$. Let $\mathbf{v}, \mathbf{v}' \in \mathbb{R}^n$ satisfy (13) and $v_j = v_j'$ for each $j \in I \backslash \{1, i\}$. For any $\mathbf{k}, \mathbf{k}' \in \mathcal{K}$, we have $\mathbf{k} \cdot \mathbf{v} > \mathbf{k}' \cdot \mathbf{v} \Leftrightarrow (1 - k_i)v_1 + k_i v_i > (1 - k_i')v_1 + k_i' v_i \Leftrightarrow (k_i' - k_i)(v_1 - v_i) > 0 \Leftrightarrow (k_i' - k_i)(v_1' - v_i') > 0 \Rightarrow \mathbf{k} \cdot \mathbf{v}' > \mathbf{k}' \cdot \mathbf{v}'$, where the third arrow holds by (13). Therefore, $k_i^*(p_1, p_i)$ depends only on $p_1 - p_i$. Moreover, by Axiom 4 and properties of $\mathcal{K}$, it takes either $\overline{k}_i$ or $\underline{k}_i$, so we have (15). Asigning $\left(\underline{k}_i, \overline{k}_i\right) = (-\alpha_i, \beta_i)$ to (14) and (15) yields EER. Condition (2) implies that $\sum_{i=2}^{n} \beta_i \leq 1$. ■

The proof mainly follows GS, whose maxmin expected utility (MEU) is characterized by Axioms 1-3 and 5:

Axiom 5. $\forall \mathbf{x}, \mathbf{y} \in D; [\forall i \in I; \mathbf{e}(x_i) \geq \mathbf{e}(y_i)] \Rightarrow \mathbf{x} \succsim \mathbf{y}$.

Axiom 5 is interpreted as Pareto efficiency in our setting. We have weakened Axiom 5 to (2) and impose Axiom 4 instead of it because an inequality averse individual reveals $(0, 0) \succ (0, 1)$ inefficiently in an ultimatum game, for example. When $n = 2$, Figures 1 and 2 depict indifference curves of MEU and EER, respectively. They differ in signs of their slopes above 45-degree line, and the difference is caused by Axioms 4 and 6.

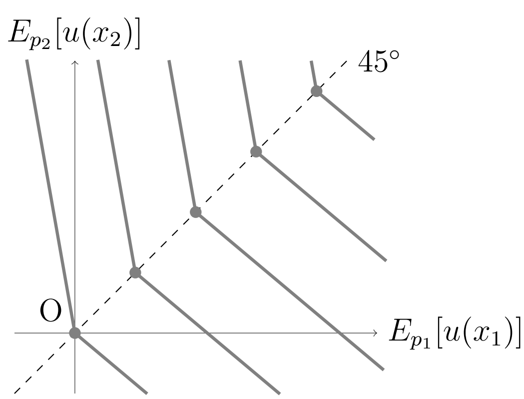

Figure 1. Indifference curves of MEU

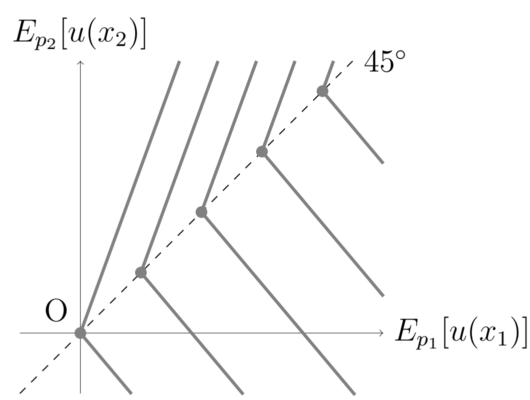

Figure 2. Indifference curves of EER

3.1 Parameter characterization

The representation (1) is said to exhibit *inequity aversion* if $\alpha_i, \beta_i > 0$. To characterize the parameters, here consider the following two axioms for each $i \in I \backslash \{1\}$.

Axiom 6. $\exists x, y, z \in \Delta Z; \mathbf{e}(y) \succ (x_i, \mathbf{e}_{-i}(y)), \mathbf{e}(y) \succ (z_i, \mathbf{e}_{-i}(y))$ and $\mathbf{e}(y) \sim \frac{1}{2}\mathbf{e}(x) \oplus \frac{1}{2}\mathbf{e}(z)$.

Axiom 7. $\exists x, y, z \in \Delta Z; \mathbf{e}(y) \succ (x_i, \mathbf{e}_{-i}(y)) \succ (z_i, \mathbf{e}_{-i}(y))$ and $\mathbf{e}(y) \sim \frac{1}{2}\mathbf{e}(x) \oplus \frac{1}{2}\mathbf{e}(z)$.

As a colollary of Theorem 1, we can characterize it as follows:

Proposition 2. Let $\succsim$ satisfy Axioms 1-4. Then, $(\alpha_i, \beta_i) \in \mathbb{R}^2_{++}$ in EER iff Axiom 6 holds. Furthermore, $\alpha_i > \beta_i$ iff Axiom 7 holds.

We can also characterize utilitarian (i.e., $\alpha_i < 0 < \beta_i$) and competitive (i.e., $\beta_i < 0 < \alpha_i$) in similar ways.

4 Conclusion

We axiomatize Fehr-Schmidt type utility representation under risk. Contrast to existing papers, our axiomatic model permits the decision maker to have a concave utility or be risk averse. Our analysis suggests that inequity aversion and uncertainty aversion have a similar behavioral foundation (i.e., the same way of violation of independence axiom). This is consistent with Kameda et al (2016) finding that individual attitudes toward distributional and uncertain tasks draw on common cognitiveneural processes.

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