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## Bayesian inference for time varying partial adjustment model with application to intraday price discovery

Kenji Hatakenaka

Kosuke Oya

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Graduate School of Economics
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

# Bayesian inference for time varying partial adjustment model with application to intraday price discovery[1]

Kenji Hatakenaka[2], Kosuke Oya[3]

**Abstract**

Price discovery is an important built-in function of financial markets and the central issue in the market microstructure research. Market participants need to know whether the price discovery has been achieved or how much progress has been made in order to trade at an appropriate price they consider. Since various economic events such as earnings announcement affect the price discovery, the intraday transition of price discovery varies date-by-date. In this study, we propose a statistical method to see when and how fast the intraday price discovery progresses using the high frequency price series on a daily basis. The proposed method consists of estimating three candidate models which gauge the different types of price discovery progress, i.e. no progress, smooth progress and abrupt progress, and selecting the most appropriate model based on Bayesian approach. We conduct simulation analysis to assess the performance of our proposed method and confirm that the method depicts the state of price discovery appropriately. The empirical study using the Japanese stock market index shows that the proposed method well categorizes the intraday price discovery progresses on a daily basis.

*Keywords:* pre-opening period, market microstructure, partial adjustment model
*JEL:* C11, G14

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[2]Graduate School of Economics, Osaka University Email:k.hatakenaka.o@icloud.com
[3]Graduate School of Economics, Osaka University Email:oya@econ.osaka-u.ac.jp

## 1. Introduction

Price discovery is an important built-in function of financial markets and the central issue in the market microstructure research. More specifically, the price discovery brings prices of assets traded on markets closer to their fundamental values by reflecting all publicly available information. Market participants need to know whether the price discovery has been achieved or how much progress has been made in order to trade at an appropriate price they consider. In this paper, we propose a statistical method to see when and how fast price discovery progresses.

Biais et al. (1999) is well known previous research about price discovery during the preopening period which is the period before the regular market session. They gauge the state of price discovery at specific time by estimating the slope coefficient of unbiasedness regression in which they regress the close-to-close return onto the return from the closing price of previous trading day to the price at the specific time. In their analysis, the slope coefficients at specific time $t$ is obtained by one unbiasedness regression using the 19 trading days of French stock market data between October 29 and November 26, 1991. They figure out the path of slope coefficients for each minute during the period from 9:30 to 12:00 which includes the opening at 10:00 am in the Paris Bourse. However, not only does their approach ignore the time series properties of the market data, but they also cannot have sufficient sample size when our interest focuses on the economic event such as quarterly earnings announcement since there are only four announcements in a year.

In a different way than Biais et al. (1999), we adopt a partial price adjustment model which focuses on the difference between the fundamental value and the observed market price of asset. The partial price adjustment model is often used in the research field of market microstructure. For example, Amihud and Mendelson (1987) conduct the empirical comparison of price behavior under the different two trading mechanisms by using the partial price adjustment model. It is worth noting that the magnitude of adjustment coefficient of the model represents the state of price discovery. We extend the partial adjustment model to models that allow time-varying adjustment coefficient using the mechanisms of the smooth transition as in Teräsvirta (1994) and the threshold coefficient as in Tong (1978) to capture how the price discovery progress.

In this paper, we consider the partial price adjustment models whose adjustment coefficient is constant, smoothly time varying, and switching abruptly. The estimation of unknown parameters is not easy due to the identification problems inherent in the smooth transition model and the threshold model. Following the previous study by Gerlach and Chen (2008), Bayesian approach is used to estimate the proposed models in this paper. For the model selection, we employ the deviance information criterion of Spiegelhalter et al. (2002) and the marginal likelihood method of Chib (1995) and Chib and Jeliazkov (2001), and evaluate these selection methods through the simulation studies as in So and Chan (2014).

The rest of this paper is organized as follows. We describe three types of partial adjustment model with an adjustment coefficient which is constant, smoothly time varying, and switching abruptly in Section 2. Section 3 provides the estimation method of three candidate models and the model selection criteria. After that, we conduct simulation analysis in Section 4 to assess the performance of our proposed method and give an empirical study using actual observed financial data in Section 5. We offer concluding remarks in Section 6.

## 2. Model description of partial adjustment models

We see how price discovery progresses by estimating three candidate models and selecting most appropriate model among them. In this section, we introduce three candidate models and describe the identification problems related to them.

### *2.1. Constant adjustment model*

First, we introduce a partial adjustment model used in Amihud and Mendelson (1987). They analyze the price behaviors of the stocks on New York Stock Exchange with followings

$$\Delta p_t = g(m_t - p_{t-1}) + u_t, \tag{1}$$

$$m_t = m_{t-1} + e_t \tag{2}$$

where $p_t$ is the logarithmic market price of asset at time $t$ and $\Delta p_t \equiv p_t - p_{t-1}$ stands for the return. Here, $m_t$ is the logarithmic efficient price which is given as the expectation of the fundamental value of the asset conditional on all publicly available information up to time $t$. It is noted that $m_t$ is unobservable and the specification of which the efficient price follows the random walk according to the efficient market hypothesis is conventional in market microstructure analysis (see Amihud and Mendelson (1987) and Hasbrouck (2007)). The market return noise $u_t$ and the innovation of efficient price $e_t$ are *i.i.d.* random variables with mean zero, constant variances, and zero covariance. Further we impose the normality assumption for $u_t$ and $e_t$ for simplicity. Combining (1) and (2) gives the following equation for the market price $p_t$

$$p_t = (1 - g)p_{t-1} + gm_{t-1} + u_t + ge_t.$$

The expectation of $p_t$ conditional on the information up to time $t-1$ implies that the current market price is determined between the previous market price and the efficient price on average, i.e. the market price is partially adjusted to the efficient price. The magnitude of adjustment coefficient $g$ represents the state of price discovery. For the case of $g = 1$, the current market price $p_t$ is appropriately adjusted to the efficient price on average. $g < 1$ ($> 1$) represents the under (over) reaction of traders to new information. In addition, we suppose $g \in (0, 2)$ to ensure the stationarity of the return process. Hereafter, we call this model as the constant partial adjustment model.

*2.2. Time varying adjustment model*

As in Biais et al. (1999), it is interesting to assess the change of state of price discovery through various economic events, e.g. before and after the opening period and earnings announcement. In the following, we extend the constant partial adjustment model to the two types of the time-varying partial adjustment models, which are based on the smooth transition autoregressive model proposed in Teräsvirta (1994) and the threshold autoregressive model proposed in Tong (1978). The smooth transition partial adjustment model is introduced to depict the gradual change of state of price discovery as follows

$$\Delta p_t = s_t(m_t - p_{t-1}) + u_t, \tag{3}$$

$$m_t = m_{t-1} + e_t, \tag{4}$$

$$s_t = a_1 + a_2 \left\{ 1 + \exp\left( -\gamma \frac{\tau_t - c}{\sigma_\tau} \right) \right\}^{-1} \tag{5}$$

where $\tau_t = t/T$ $(t = 2, \dots, T)$ and $\sigma_\tau$ is the standard deviation of $\tau_t$ which is used to standardization. Eq. (5) contains the logistic function of $\tau_t$ with the speed parameter $\gamma > 0$ and the location parameter $0 < c < 1$. The adjustment coefficient $s_t$ transits smoothly from $a_1$ to $a_1 + a_2$ where the parameter restrictions $a_1, a_1 + a_2 \in (0, 2)$ are imposed for non-explosive condition of $\Delta p_t$. As depicted in Figure 1, the adjustment coefficient stays at $a_1 + a_2/2$ when $\gamma = 0$, and instantaneously changes from $a_1$ to $a_1 + a_2$ at $\tau_t = c$ when $\gamma = \infty$. These are the special case of the smooth transition partial adjustment model.

We adopt the Markov chain Monte Carlo (MCMC) method for the estimation of the smooth transition model because the maximization of the log likelihood without any restrictions is difficult, specially with respect to $\gamma$. As shown in Ekner and Nejstgaard (2013), the

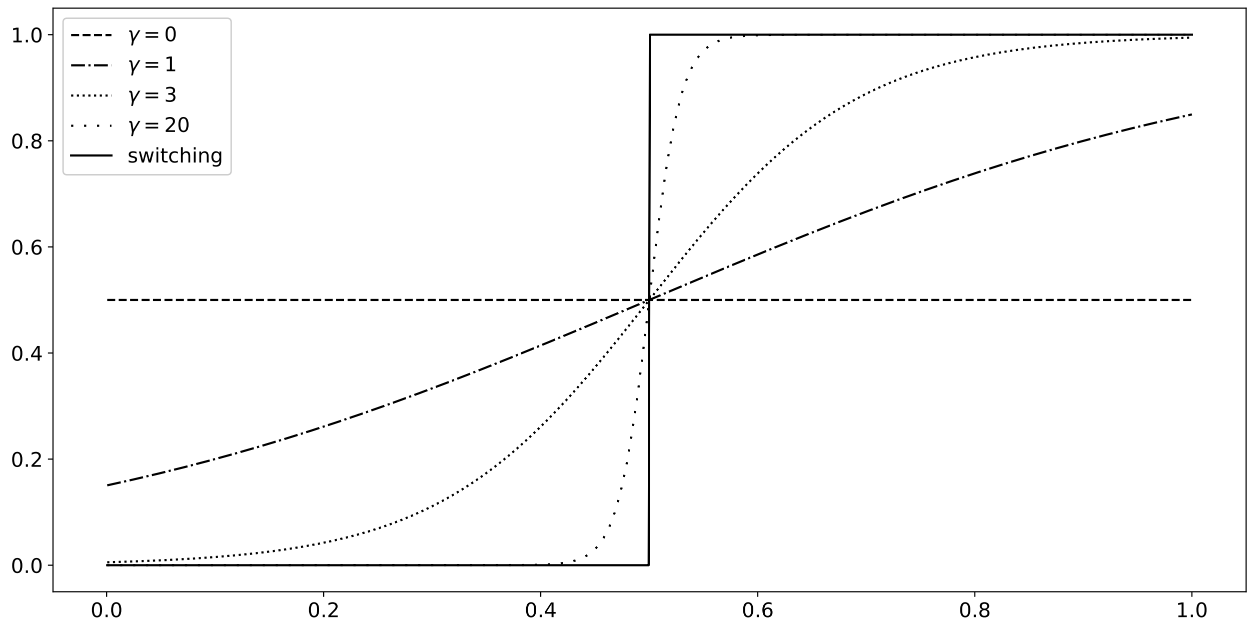


**Figure 1:** Shape of adjustment coefficients $s_t = (1 + \exp\{-\gamma(\tau_t - 0.5)/\sigma_\tau\})^{-1}$ with each speed parameter ($\gamma = 0, 1, 3, 20$) and switching adjustment coefficient with threshold $c = 0.5$. $\sigma_\tau$ is standard deviation of $\tau_t = t/T$.

first partial derivative of the log likelihood function with respect to $\gamma$ takes 0 at $\gamma = \infty$. This implies that the likelihood becomes at least locally maximum at $\gamma = \infty$ even if the true $\gamma$ is not $\infty$. Moreover, Figure 1 shows that the $\gamma$ in (5) which takes the value over 20 makes the path of adjustment coefficient by the smooth transition model virtually indistinguishable from that by the threshold model which will be given below, see also Gerlach and Chen (2008). Depending on the purpose of analysis, the accuracy of distinguishing the difference between the paths from the two different models will differ. For our purpose of accurately capturing the state of price discovery, it is not necessary to distinguish between the smooth transition model with $\gamma$ of 20 or more and the threshold model. Instead of the smooth transition model with the large value of $\gamma$, we provide the following threshold model for the case of abruptly changing $s_t$ as an alternative.

$$\Delta p_t = s_t(m_t - p_{t-1}) + u_t, \tag{6}$$

$$m_t = m_{t-1} + e_t, \tag{7}$$

$$s_t = a_1 + a_2 I_{\tau_t \geq c}, c \in \{2/T, 3/T, \ldots, 1\}. \tag{8}$$

Eq. (8) makes $s_t$ switching abruptly from $a_1$ to $a_1 + a_2$ instead of smooth transition with (5). As described above, for the case of $\gamma = \infty$, the smooth transition partial adjustment model collapses to the threshold partial adjustment model. For the threshold partial adjustment model (6)–(8), the non-explosive condition ($a_1, a_1 + a_2 \in (0, 2)$) is also necessary. Hereafter we refer (1)–(2), (3)–(5), and (6)–(8) as models $\mathcal{C}$, $\mathcal{S}$, and $\mathcal{T}$ for brevity.

*2.3. Restrictions for parameters*

We have to consider an identification problem for models $\mathcal{S}$ and $\mathcal{T}$. It is obvious that the identification problem for the parameters $\gamma$ and $c$ is caused by imposing $a_2 = 0$ on model $\mathcal{S}$ for the expression of constant adjustment coefficient and the same problem occurs for the parameter $c$ in model $\mathcal{T}$. Therefore, we impose the restriction $a_2 \neq 0$ for the exclusion of constant adjustment coefficient case from the scope of models $\mathcal{S}$ and $\mathcal{T}$.

By restricting parameters in each model and selecting appropriate model from the candidates, we can avoid the identification problems and categorize the path of adjustment coefficient into one of three different types: constant, smoothly time-varying, and switching abruptly, with respective corresponding models.

## 3. Bayesian inference

We estimate all three candidate models and carry out a model selection, using Bayesian approach. The inference based on the posterior distribution through Markov chain Monte Carlo (MCMC) is convenient to incorporate parameter restrictions described in the previous section. Further, the Bayesian approach adopted in this paper does not involve the numerical optimization unlike the maximum likelihood method which is often associated with some difficulties in maximizing the log likelihood and obtaining the standard error of estimators for the inference. In following, we provide the Bayesian estimation and model selection in detail.

*3.1. Prior distributions*

We first introduce the prior distributions for the speed parameter $\gamma$ and the location parameter $c$ of model $\mathcal{S}$. For the reason mentioned in subsection 2.2, we adopt the truncated normal prior for $\log\gamma$, whose mean $\mu_\gamma$, variance $\sigma_\gamma^2$, and the lower and upper bounds $\log\ell_\gamma$ and $\log u_\gamma$, that is, $\log\gamma \sim TN(\mu_\gamma, \sigma_\gamma^2)_{(\log\ell_\gamma, \log u_\gamma)}$ to distinguish between model $\mathcal{S}$ and model $\mathcal{T}$. For the location parameter $c$ in model $\mathcal{S}$, we set $c \sim TN(\mu_c, \sigma_c^2)_{(\ell_c, u_c)}$ as the prior distribution. For the case of model $\mathcal{T}$, we use the discrete uniform prior for $c$ with the same lower and upper bounds $\ell_c$ and $u_c$ as used for model $\mathcal{S}$, i.e. $\ell_c < u_c$ and $\ell_c, u_c \in \{2/T, 3/T, \ldots, 1\}$. In addition, as we mentioned earlier, we need to restrict the range of adjustment coefficient for the non-explosiveness of return process and identification problems in models $\mathcal{S}, \mathcal{T}$. So we set the following truncated normal prior with the positive lower bound: $a_1 \sim TN(\mu_{a_1}, \sigma_{a_1}^2)_{(\ell_{a_1}, 2)}$. $a_2$ needs to satisfy $a_2 \in (-a_1, 2-a_1)$ for the non-explosive condition and $a_2 \neq 0$ for the identification problem. Because we focus on how the price discovery progress, i.e. the non-decreasing adjustment coefficient case in this paper, we assume the conditional truncated normal prior for $a_2$ as follow $a_2|a_1 \sim TN(1-a_1, \sigma_{a_2}^2)_{(\ell_{a_2}, 2-a_1)}$ where $\ell_{a_2} > 0$. For $g$ which represents the state of price discovery in model $\mathcal{C}$, similar to $a_1$ and $a_2$, we adopt the same prior as $a_1$. For the noise variance $\sigma_u^2$, we use the inverse gamma distribution with shape parameter $\alpha_u$ and rate parameter $\beta_u$, that is $IG(\alpha_u, \beta_u)$, as the conjugate prior in the simulation study and the uniform prior $U(\ell_u, u_u)$ in the empirical study. For the sampling procedure of $\sigma_m^2$ which is the innovation variance of unobservable efficient price $m_t$, we conduct ancillarity-sufficiency interweaving strategy (ASIS) proposed in Yu and Meng (2011). We will give the prior distribution and the sampling procedure of $\sigma_m^2$ in the following subsection. Overall, we assume that the prior distributions for each parameter are independent each other except for $a_1$ and $a_2$. The detail of sampling procedure is described in following.

*3.2. Sampling procedure*

Firstly, we describe the sampling procedure for model $\mathcal{S}$. Denote $\Theta$ the parameter set of the target model, here $\Theta = (\gamma, c, a_1, a_2, \sigma_m^2, \sigma_u^2)$.

step 0. Determine $\Theta^{(0)}$ and set $i = 1$.

step 1. Draw the state variables $\widetilde{m}_T^{(i)} = (m_2^{(i)}, m_3^{(i)}, \cdots, m_T^{(i)})$.

step 2. Draw individually each parameter in $\Theta^{(i)}$.

step 3. Repeat step1 and step2 for $i = 1, \ldots, N$.

In estimating the other candidate models, we similarly use the same procedure for the other parameter set in those models.

As for step 1, by combining with the Carter and Kohn (1994) algorithm, Gibbs sampling is possible to obtain a draw for the unobservable efficient price $\widetilde{m}_T$.

As for the sampling method of $\gamma$ and $c$ in model $\mathcal{S}$, we implement the Metropolis-Hastings (MH) algorithm. In drawing proposals of $\gamma$ and $c$, we adopt the random walk MH algorithm during burn-in period (first $M$ iterations) and the adaptive MH algorithm after burn-in period (another $M^*$ iterations). The proposal mean and variance of the adaptive MH algorithm are chosen to be the sample mean and variance for the last quarter of burn-in period. As for the random walk MH algorithm, we use the following proposal distribution for $\gamma$

$$\gamma|\gamma^{i-1} \sim G\left(\frac{\Delta_\gamma}{\gamma^{i-1}}, \frac{\Delta_\gamma}{(\gamma^{i-1})^2}\right)_{(\ell_\gamma, u_\gamma)} \tag{9}$$

where $G(\alpha, \beta)_{(a,b)}$ denotes a truncated gamma distribution with shape parameter $\alpha$, rate parameter $\beta$, lower bound $a$, and upper bound $b$, and $\Delta_\gamma$ is the pre-determined constant for variance of $\gamma$. For $c$ of model $\mathcal{S}$, we conduct following sampling method to restrict the parameter space of $c$. Firstly, we draw a random variable $x^*$ from $N(\mu_x, \sigma_x^2)$ and perform the following transformation to generate $c^*$,

$$c = \frac{\exp(x)}{1 + \exp(x)} (u_c - \ell_c) + \ell_c, \ c \in (\ell_c, u_c).$$

After the burn-in period, random samples of $\gamma$ and $x$ are obtained from the updated proposal distribution.

As for the sampling of the other parameters ($c$ in model $\mathcal{T}$, $g$, $a_1$, $a_2$, $\sigma_m^2$, and $\sigma_u^2$), Gibbs sampling is adopted. Using the conjugate prior distribution described in the previous section, we can easily obtain samples of the parameters $a_1$, $a_2$, $g$, and $\sigma_u^2$ by Gibbs sampling. The full conditional posterior distributions of $a_1$, $a_2$, and $g$ are the truncated normal distribution and that of $\sigma_u^2$ is the inverse gamma distribution. Note that the posterior distribution of $\sigma_u^2$ is the truncated inverse gamma distribution when we adopt the uniform prior as $\sigma_u^2 \sim U(\ell_u, u_u)$. As for $c$ in model $\mathcal{T}$, we employ the method to generate a change point as in Carlin et al. (1992).

For the estimation of $\sigma_m^2$ in all candidate models, we conduct ASIS to improve the sampling efficiency. By setting $m_t = m_t^* \sigma_m$ and $e_t = e_t^* \sigma_m$, we can reparameterize models $\mathcal{C}$, $\mathcal{S}$, and $\mathcal{T}$. For example, models $\mathcal{S}$ and $\mathcal{T}$ can be rewritten as

$$\Delta p_t = s_t(m_t^* \sigma_m - p_{t-1}) + u_t, \tag{10}$$

$$m_t^* = m_{t-1}^* + e_t^* \tag{11}$$

where $\sigma_m$ is shifted from the state equation to the observation equation. The model $\mathcal{C}$

in another parameterization is obtained by setting $s_t = g$ in (10). The full conditional posterior distribution of $\sigma_m$ is a truncated normal by setting the prior density as $\sigma_m \sim TN(\mu_0, \sigma_0^2)_{(0,\infty)}$. In the parameterization utilized in the previous section, the full conditional posterior distribution of $\sigma_m^2$ is an inverse gamma distribution when we use an inverse gamma distribution as the prior whereas a truncated inverse gamma distribution using a uniform prior. The ASIS we adopt for $\sigma_m^2$ is the following

Step 1. Calculate $\widetilde{m}_T^* = \widetilde{m}_T / \sigma_m$. Here we consider $\Theta$ contains $\sigma_m$ instead of $\sigma_m^2$. Draw $\sigma_m$ from

$$\sigma_m | \widetilde{p}_T, \widetilde{m}_T^*, \Theta_{-\sigma_m} \sim TN(\mu_1, \sigma_1^2)_{(0,\infty)}, \quad (\textit{Ancillary})$$
$$\mu_1 = \frac{\sum_{t=2}^T (s_t m_t^*)(\Delta p_t + s_t p_{t-1})\sigma_0^2 + \mu_0 \sigma_u^2}{\sum_{t=2}^T (s_t m_t^*)^2 \sigma_0^2 + \sigma_u^2}, \sigma_1^2 = \frac{\sigma_0^2 \sigma_u^2}{\sum_{t=2}^T (s_t m_t^*)^2 \sigma_0^2 + \sigma_u^2}$$

where $\widetilde{p}_T = (p_1, \ldots, p_T)$, $s_t$ is given by (5) for model $\mathcal{S}$, (8) for model $\mathcal{T}$ and $g$ for model $\mathcal{C}$, and $\mu_0, \sigma_0^2$ are the hyper-parameters of the prior distribution $TN(\mu_0, \sigma_0^2)_{(0,\infty)}$ for $\sigma_m$.

Step 2. Calculate $\widetilde{m}_T = \widetilde{m}_T^* \sigma_m$ and draw $\sigma_m^2$ from

$$\sigma_m^2 | \widetilde{p}_T, \widetilde{m}_T, \Theta_{-\sigma_m^2} \sim IG(\alpha_1, \beta_1), \quad (\textit{Sufficient})$$
$$\alpha_1 = \frac{T-1}{2} + \alpha_m, \quad \beta_1 = \beta_m + \frac{1}{2} \sum_{t=3}^T (m_t - m_{t-1})^2$$

where $\alpha_m, \beta_m$ are the hyper-parameters of the prior distribution $IG(\alpha_m, \beta_m)$ for $\sigma_m^2$. When we set the prior density as $\sigma_m^2 \sim U(\ell_m, u_m)$, draw from the truncated inverse gamma posterior with shape parameter $\alpha_1$ and rate parameter $\beta_1$ where $\alpha_m = -1, \beta_m = 0$.

For detail of ASIS, see Yu and Meng (2011) and Kastner and Frühwirth-Schnatter (2014).

In order to select the appropriate path type of adjustment coefficient, we conduct the

Bayesian model selection. The model selection criteria we adopt are Deviance Information Criterion (DIC) of Spiegelhalter et al. (2002) and the marginal likelihood method of Chib (1995) and Chib and Jeliazkov (2001). We will evaluate these selection criteria in the following simulation analysis.

## 4. Simulation analysis

We are interested in whether the selection criteria provide an appropriate model with respect to the transition type of adjustment coefficient and the signal to noise ratio for the observed data. In this section, we conduct a simulation analysis to see how our proposed estimation method and model selection criteria work.

### *4.1. Settings for estimation and simulation*

For the estimation of three candidate models, we apply the sampling scheme described in the previous section. We employ $M = 10000$ iterations as burn-in and another $M^* = 15000$ iterations for posterior sampling of parameters and calculating DIC. Chib's marginal likelihood requires another 5500 iterations where we discard the first 500 iterations as burn-in. In the calculation of Chib's marginal likelihood of the model $\mathcal{S}$, we conduct the same adaptive sampling method as in the estimation of the model $\mathcal{S}$.

The simulation data is generated from the data generating process (3) and (4) in various settings for the transition speed $\gamma$ and the signal to noise ratio $\nu \equiv \sigma_m^2/\sigma_u^2$. The speed parameter is set as $\gamma = 0$ ($s_t$ is constant), $\gamma = 5, 15$ ($s_t$ is smoothly time varying), and $\gamma = 150, \infty$ ($s_t$ switches abruptly). Because model $\mathcal{S}$ collapses to model $\mathcal{T}$ when $\gamma = \infty$, we generate the simulated data using model $\mathcal{T}$ in the case of $\gamma = \infty$. The signal to noise ratio is set as $\nu = 5, 10$ (efficient case, which is the case where the information for efficient price predominates at the observed price) and $\nu = 1$ (noisy case). We provide the simulated data with 15 settings with $\gamma = 0, 5, 15, 150, \infty$ and $\nu = 1, 5, 10$. We set other parameters as

**Table 1:** Model selection result using Bayes Factor and DIC

| | | $\nu = 1$ | | $\nu = 5$ | | $\nu = 10$ | |
|---|---|---|---|---|---|---|---|
| model type | setting of $\gamma$ | BF | DIC | BF | DIC | BF | DIC |
| $\mathcal{C}$ | 0 | 98 | 49 | 99 | 37 | 99 | 44 |
| $\mathcal{S}$ | 5 | 80 | 75 | 100 | 100 | 100 | 100 |
| $\mathcal{S}$ | 15 | 30 | 52 | 89 | 93 | 97 | 99 |
| $\mathcal{T}$ | 150 | 96 | 93 | 98 | 96 | 98 | 97 |
| $\mathcal{T}$ | $\infty$ | 97 | 92 | 100 | 100 | 100 | 100 |

Figures in table are the proportion (in %) of correct selection. 100 replications are conducted for each setting. BF in this table represents Bayes Factor calculated by Chib's method.

$c = 0.5, a_1 = 0.1, a_2 = 0.9, \sigma_u^2 = 0.001$, and initial value of log price $p_1 = \log 5000$ and log efficient price $m_1 = \log 5050$. The sample size of simulated data is set to 1800, assuming 30 minutes observations in seconds and the log price series multiplied by 100 is used to display the return in percent. We generate 100 replications for all settings of $\gamma$ and $\nu$.

For the settings of boundary, we set $\ell_\gamma = 1$, $u_\gamma = 20$, $\ell_c = 0.1$, $u_c = 0.9$, $\ell_g = \ell_{a_1} = 0.01$, $\ell_{a_2} = 0.05$. For the prior distributions, we set $\log\gamma \sim TN(10, 3^2)_{(0,\log 20)}$, $c \sim TN(0.45, 0.5^2)_{(0.1,0.9)}$ (for the model $\mathcal{S}$), $g \sim TN(0.4, 5^2)_{(0.01,2)}$, $a_1 \sim TN(0.4, 5^2)_{(0.01,2)}$, $a_2|a_1 \sim TN(1 - a_1, 5^2)_{(0.05,2-a_1)}$, $\sigma_u^2$, $\sigma_m^2 \sim IG(10^{-3}, 10^{-3})$, $\sigma_m \sim TN(0, 3^2)_{(0,\infty)}$. For the hyper-parameters of the proposal distributions, we set $\Delta_\gamma = 2.5^2, \sigma_x^2 = 0.03^2$. In the following, we present the results of model selection and estimation for each setting of simulation.

### *4.2. Model selection result*

First of all, we define the correct model selection in this simulation study as follows: the selection is correct when the model $\mathcal{T}$ is selected in the case of $\gamma = 150, \infty$, the model $\mathcal{S}$ is selected in the case of $\gamma = 5, 15$, and the model $\mathcal{C}$ is selected in the case of $\gamma = 0$. In the following, we will compare the model selection criteria in terms of the correct selection ratio. To assess the statistical inference proposed in this paper, we conduct the estimation and model selection iteratively 100 times with the simulated data for each parameter setting. Table 1 shows the ratio of correct selection out of 100 replications. In the efficient case ($\nu = 5, 10$), Chib's method performs well for all settings of $\gamma$ while DIC does not for the case of $\gamma = 0$ (the model $\mathcal{C}$ is correct). In the noisy case ($\nu = 1$), Chib's method is less accurate for the case of $\gamma = 5, 15$ (the model $\mathcal{S}$ is correct) while DIC is also less accurate for the case of $\gamma = 0, 5, 15$. We will see the case of $(\gamma, \nu) = (15, 1)$ in detail below.

For convenience sake, we denote Bayes factor in favor of model $\mathcal{M}_0$ and against model $\mathcal{M}_1$ as $\mathrm{BF}(\mathcal{M}_0/\mathcal{M}_1)$. We select the model $\mathcal{M}_0$ when $\log \mathrm{BF}(\mathcal{M}_0/\mathcal{M}_1) > 0$. The first and second rows of Figure 2 depict boxplots of $\log \mathrm{BF}(\text{correct model}/\text{incorrect model})$. Boxplots of $\log \mathrm{BF}(\mathcal{C}/\mathcal{S})$ and $\log \mathrm{BF}(\mathcal{C}/\mathcal{T})$ are omitted except for the case of $\gamma = 0$ because none of two criteria select the model $\mathcal{C}$ when $\gamma > 0$. The boxplot of $\log \mathrm{BF}(\mathcal{S}/\mathcal{T})$ for $\gamma = 15$ indicates that model $\mathcal{T}$ tends to be incorrectly selected instead of model $\mathcal{S}$. In general, model $\mathcal{S}$ and model $\mathcal{T}$ become indistinguishable as the observed price data become noisy.

The third and fourth rows of Figure 2 provide the boxplots of difference of DICs for the correct model and the incorrect model, that is $\mathrm{DIC}_{\text{correct}} - \mathrm{DIC}_{\text{incorrect}}$. We select model $\mathcal{M}_0$ against model $\mathcal{M}_1$ when $\mathrm{DIC}_{\mathcal{M}_0} - \mathrm{DIC}_{\mathcal{M}_1} < 0$. The boxplots for DIC show the same tendency as the selection using the Bayes factor for the case of less informative observed price data. Moreover, selection using DIC differences does not clearly distinguish between model $\mathcal{C}$ and other model types when the model $\mathcal{C}$ is correct whereas the selection based on Bayes factor does.

Regarding DIC vs Chib's marginal likelihood approach, the performances of both criteria are almost the same or Chib's marginal likelihood approach is superior to DIC except for the case of $(\gamma, \nu) = (15, 1)$. Although neither method makes a clear distinction between model $\mathcal{S}$ and model $\mathcal{T}$ when we use the noisy data, Chib's method can distinguish between model $\mathcal{C}$ and other model types more accurately than DIC does. From those results, Chib's method is suitable for our purpose. We will use Chib's method to select appropriate model for empirical study in the next section.

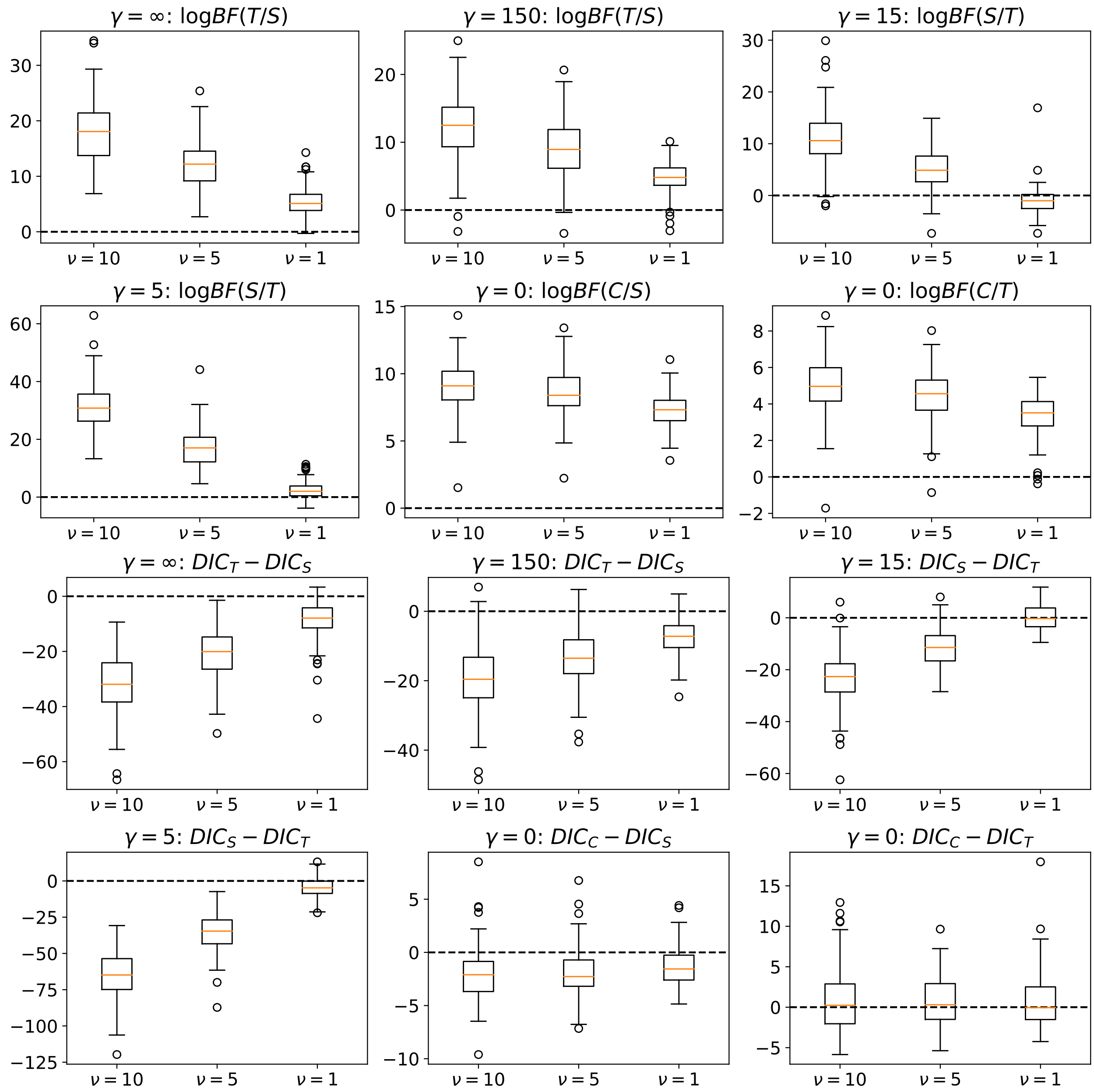


**Figure 2:** Boxplots of log of the Bayes factor calculated by Chib's method and the difference of DIC for each setting of $\gamma$ and $\nu$. The lower whisker is at the lowest data above $Q_1 - 1.5(Q_3 - Q_1)$ and the higher whisker is at the highest data below $Q_3 + 1.5(Q_3 - Q_1)$ where $Q_i$ is $i$-th quartile. The points which are outside the whiskers are defined as outliers and they are plotted as individual points.

### *4.3. Estimation result of the adjustment coefficient and parameters*

We also report the estimation result of $s_t$ and all parameters. For brevity, $\hat{s}_t$ denotes the estimate of the adjustment coefficient, that is, $\hat{s}_t = \hat{a}_1 + \hat{a}_2(1 + \exp\{-\hat{\gamma}(\tau_t - \hat{c})/\sigma_\tau\})^{-1}$ in

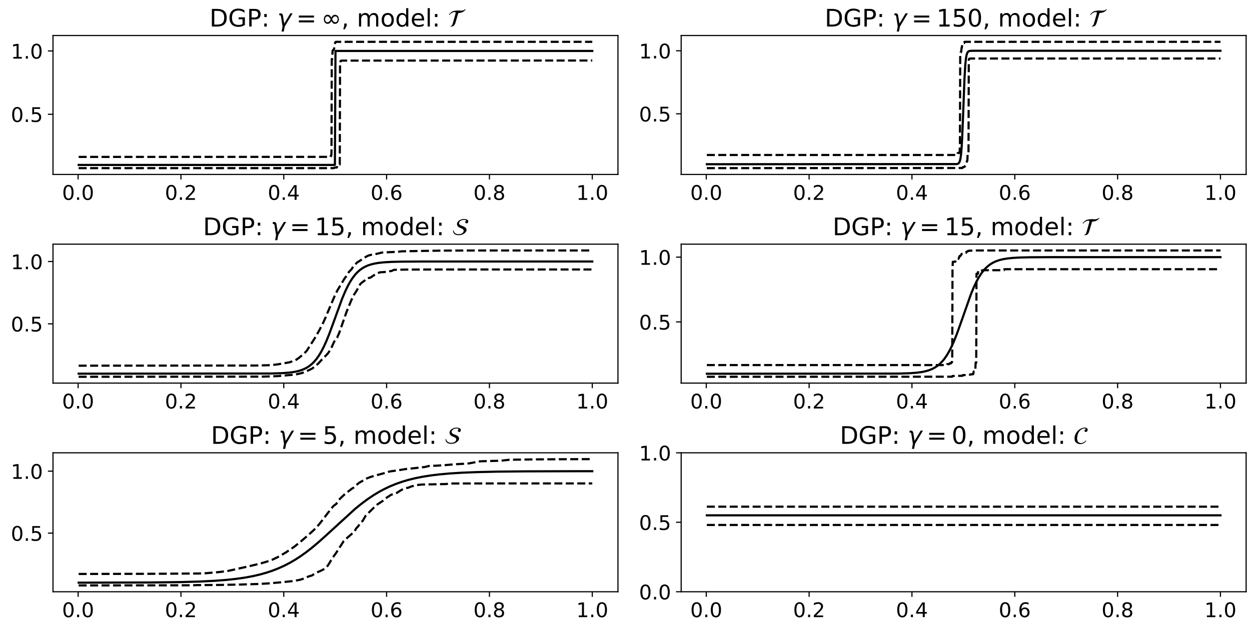


**Figure 3:** $s_t = 0.1+0.9(1+\exp\{-\gamma(\tau_t-0.5)/\sigma_\tau\})^{-1}$ for the simulation data with $\gamma = 0, 5, 15, 150, \infty$ when $\nu = 1$ (noisiest case). For each figure, true $s_t$ is plotted by solid line and 5 and 95 percentiles of $\hat{s}_t$ in 100 replications are plotted by the upper and lower dotted lines.

model $\mathcal{S}$ , $\hat{s}_t = \hat{a}_1 + \hat{a}_2 I_{\tau_t \geq \hat{c}}$ in model $\mathcal{T}$, and $\hat{s}_t = \hat{g}$ in model $\mathcal{C}$ ($\hat{\theta}$ represents a posterior mean estimate of $\theta$). We have similar estimation results of $s_t$ in all cases of $\nu$. So we report only for the noisiest case ($\nu = 1$). Figure 3 represents the result of $\hat{s}_t$ for each setting of $\gamma$. For each setting, we plot true $s_t$ and 5 and 95 percentiles of $\hat{s}_t$ obtained by 100 replications. In Figure 3, the most selected model in 100 replications is denoted as "selected model". Only for the case of $\gamma = 15$, the selected model differs from the model that provides the simulated data. Overall, the interval between 5 and 95 percentiles of $\hat{s}_t$ by the selected model is moderately narrow and contains true $s_t$ including the case of $\gamma = 15$.

Finally, we provide the summary statistics of posterior mean of parameters for 100 replications in Table 2. Similar to $\hat{s}_t$, we report the estimation results of models $\mathcal{S}$ and $\mathcal{T}$ only for the case of $(\gamma, \nu) = (15, 1)$. For the remaining easily distinguishable cases ($\gamma = 0, 5, 150, \infty$), we report only the results of the model selected by Chib's method, which is a correct model. The selected model name is below the setting of $\gamma$ in Table 2.

**Table 2:** Mean and standard deviation of the posterior mean for each parameter in 100 replications

$\nu = 1$

| | $\gamma = 0$ model: $\mathcal{C}$ | | $\gamma = 5$ model: $\mathcal{S}$ | | $\gamma = 15$ model: $\mathcal{S}$ | | $\gamma = 15$ model: $\mathcal{T}$ | | $\gamma = 150$ model: $\mathcal{T}$ | | $\gamma = \infty$ model: $\mathcal{T}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Mean | Std. | Mean | Std. | Mean | Std. | Mean | Std. | Mean | Std. | Mean | Std. |
| $\gamma$ | | | 7.7064 | 3.0786 | 14.0650 | 2.3390 | | | | | | |
| $c$ | | | 0.5041 | 0.0241 | 0.5003 | 0.0118 | 0.5036 | 0.0172 | 0.5007 | 0.0046 | 0.5001 | 0.0044 |
| $a_1$ | | | 0.1106 | 0.0304 | 0.1120 | 0.0261 | 0.1148 | 0.0266 | 0.1145 | 0.0349 | 0.1066 | 0.0262 |
| $a_2$ | | | 0.8848 | 0.0629 | 0.8991 | 0.0490 | 0.8673 | 0.0462 | 0.8908 | 0.0426 | 0.8943 | 0.0461 |
| $g$ | 0.5503 | 0.0428 | | | | | | | | | | |
| $\sigma_u^2$ | 0.0010 | 0.0001 | 0.0010 | 0.0001 | 0.0010 | 0.0001 | 0.0010 | 0.0001 | 0.0010 | 0.0001 | 0.0010 | 0.0001 |
| $\sigma_m^2$ | 0.0010 | 0.0001 | 0.0010 | 0.0001 | 0.0010 | 0.0001 | 0.0010 | 0.0001 | 0.0010 | 0.0001 | 0.0010 | 0.0001 |
| $\nu$ | 1.0259 | 0.1595 | 1.0131 | 0.1537 | 1.0444 | 0.1188 | 1.0074 | 0.1206 | 1.0150 | 0.1295 | 1.0323 | 0.1198 |

$\nu = 5$

| | $\gamma = 0$ model: $\mathcal{C}$ | | $\gamma = 5$ model: $\mathcal{S}$ | | $\gamma = 15$ model: $\mathcal{S}$ | | $\gamma = 150$ model: $\mathcal{T}$ | | $\gamma = \infty$ model: $\mathcal{T}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Mean | Std. | Mean | Std. | Mean | Std. | Mean | Std. | Mean | Std. |
| $\gamma$ | | | 5.5829 | 1.1481 | 14.9304 | 2.1573 | | | | |
| $c$ | | | 0.5019 | 0.0135 | 0.5007 | 0.0063 | 0.5000 | 0.0024 | 0.4999 | 0.0013 |
| $a_1$ | | | 0.1055 | 0.0273 | 0.1022 | 0.0222 | 0.1054 | 0.0202 | 0.1015 | 0.0222 |
| $a_2$ | | | 0.8965 | 0.0417 | 0.9036 | 0.0346 | 0.8982 | 0.0323 | 0.8974 | 0.0333 |
| $g$ | 0.5360 | 0.0588 | | | | | | | | |
| $\sigma_u^2$ | 0.0011 | 0.0002 | 0.0010 | 0.0001 | 0.0010 | 0.0001 | 0.0010 | 0.0001 | 0.0010 | 0.0001 |
| $\sigma_m^2$ | 0.0050 | 0.0004 | 0.0050 | 0.0004 | 0.0050 | 0.0004 | 0.0050 | 0.0004 | 0.0051 | 0.0004 |
| $\nu$ | 5.2214 | 1.2223 | 5.0947 | 0.6374 | 5.0716 | 0.4996 | 5.0104 | 0.4995 | 5.1290 | 0.4883 |

$\nu = 10$

| | $\gamma = 0$ model: $\mathcal{C}$ | | $\gamma = 5$ model: $\mathcal{S}$ | | $\gamma = 15$ model: $\mathcal{S}$ | | $\gamma = 150$ model: $\mathcal{T}$ | | $\gamma = \infty$ model: $\mathcal{T}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Mean | Std. | Mean | Std. | Mean | Std. | Mean | Std. | Mean | Std. |
| $\gamma$ | | | 5.2748 | 0.8084 | 15.1396 | 2.0692 | | | | |
| $c$ | | | 0.5014 | 0.0110 | 0.5006 | 0.0050 | 0.4993 | 0.0020 | 0.5000 | 0.0010 |
| $a_1$ | | | 0.1022 | 0.0212 | 0.1003 | 0.0155 | 0.1032 | 0.0156 | 0.1009 | 0.0159 |
| $a_2$ | | | 0.9005 | 0.0371 | 0.9040 | 0.0314 | 0.8990 | 0.0305 | 0.8972 | 0.0294 |
| $g$ | 0.5327 | 0.0464 | | | | | | | | |
| $\sigma_u^2$ | 0.0011 | 0.0003 | 0.0010 | 0.0001 | 0.0010 | 0.0001 | 0.0010 | 0.0001 | 0.0010 | 0.0001 |
| $\sigma_m^2$ | 0.0102 | 0.0008 | 0.0101 | 0.0009 | 0.0100 | 0.0009 | 0.0100 | 0.0008 | 0.0102 | 0.0007 |
| $\nu$ | 10.5577 | 2.8886 | 10.1786 | 1.2412 | 10.1080 | 0.9981 | 10.0019 | 0.9654 | 10.2641 | 0.9746 |

$\nu = \sigma_m^2/\sigma_u^2$ and the true values of parameters are $a_1 = 0.1, a_2 = 0.9, c = 0.5, g = 0.55, \sigma_u^2 = 0.001$.

For all settings of $\gamma$, true values of all parameters are included within two standard deviations of their means. The posterior mean of $\gamma$ tends to deviate slightly from the true value. However, for our purpose, what is important is that the path of estimated $\hat{s}_t$ is close to the true $s_t$, not whether the estimate of $\gamma$ is highly accurate.

In fact, we have selected the incorrect model $\mathcal{T}$ instead of the correct model $\mathcal{S}$ for the case of $(\gamma, \nu) = (15, 1)$. But the implication from the estimated path of $s_t$ is almost the same as that from the true path of $s_t$ for the following reasons. In this case, the true start and end points of change in the adjustment coefficient are very close, the estimated change point $\hat{c}$ by the incorrectly selected model $\mathcal{T}$ is in between and the parameters $a_1$ and $a_2$ are correctly estimated.

## 5. Empirical analysis

We apply our proposed method to the financial time series data observed in Tokyo Stock Exchange (TSE) to examine how the intraday price adjustment during the preopening period varies on a daily basis. The data for this study is TOPIX Core30 which is a market index composed of the 30 issues with the highest market capitalization and liquidity in the first section of TSE. The sampling frequency of the data we use is one second and the sample period is 15 minutes before and after the opening auction at 9:00 am, i.e. from 8:45 to 9:15, on the five trading days of the second week of June, 2018.

The same prior distributions are adopted as in subsection 4.1 except for $\sigma_u^2$ and $\sigma_m^2$. We use a uniform prior $U(0, 1)$ for $\sigma_m^2$ and $\sigma_u^2$. The settings of iterations for the MCMC sampling and obtaining the marginal likelihood for each model are also the same as in subsection 4.1.

Our proposed method clearly categorized the intraday price discovery progresses on a daily basis as shown in Table 3. Table 4 summarizes the posterior samples of parameters for the selected model on each day. All the absolute values of CD are smaller than 2 except $a_1$ for June 4 and the inefficient factor ranges from 1.245 to 71.989, suggesting that the

**Table 3:** Result of model selection for TOPIX Core30

| | Model | Log of marginal likelihood | S.E. |
|---|---|---|---|
| June 4, 2018 | $\mathcal{S}$ | 5961.97 | 0.1535 |
| $\mathcal{T}$ is selected | $\mathcal{T}$ | 6114.26 | 0.0344 |
| | $\mathcal{C}$ | 5981.95 | 0.1901 |
| June 5, 2018 | $\mathcal{S}$ | 7135.72 | 0.0558 |
| $\mathcal{T}$ is selected | $\mathcal{T}$ | 7153.39 | 0.0440 |
| | $\mathcal{C}$ | 6989.66 | 0.1336 |
| June 6, 2018 | $\mathcal{S}$ | 6694.01 | 0.1773 |
| $\mathcal{T}$ is selected | $\mathcal{T}$ | 6992.50 | 0.0305 |
| | $\mathcal{C}$ | 6597.38 | 0.1610 |
| June 7, 2018 | $\mathcal{S}$ | 6913.05 | 0.0451 |
| $\mathcal{S}$ is selected | $\mathcal{T}$ | 6741.30 | 0.0718 |
| | $\mathcal{C}$ | 6712.25 | 0.1295 |
| June 8, 2018 | $\mathcal{S}$ | 2662.85 | 0.0838 |
| $\mathcal{C}$ is selected | $\mathcal{T}$ | 2683.93 | 0.0933 |
| | $\mathcal{C}$ | 2705.77 | 0.0945 |

sampling algorithm adopted in this study works well. As a point to note, CD and Inef for June 4, 5 and 6 for the change point $c$ of model $\mathcal{T}$ were not available because the posterior sample distributions were degenerated. At these change points, the observed data clearly changed their distributions as depicted in Figure 4. As the model $\mathcal{T}$, i.e. eqs. (6)–(8), suggests a high adjustment coefficient $a_1 + a_2$ contributes to increase the observed return variance relative to the return variance when the adjustment coefficient is $a_1$.

The estimated paths of adjustment coefficient in Figure 5 show that how the price discovery works is also various for each day. We find that the improvements in the price informational efficiency were observed on June 4, 5, 6, and 7, and the progresses of the first three were sharp and the last one was gradual. The adjustments started to change just before the market opening period of June 5, 6, and 7. In contrast, the information efficiency for June 8, 2018 was low and constant even before the market opened, and remained constant after the market opened. It is worth noting that all of the estimated price adjustments were

**Table 4:** Summaries of posterior sample

| Date | | Mean | Std. | CI95 lower | Median | CI95 upper | CD | Inef. |
|---|---|---|---|---|---|---|---|---|
| June 4, 2018 | $c$ | 0.108 | 0.000 | 0.108 | 0.108 | 0.108 | NA | NA |
| model $\mathcal{T}$ | $a_1$ | 0.012 | 0.002 | 0.010 | 0.011 | 0.016 | 2.908 | 2.146 |
| | $a_2$ | 0.581 | 0.028 | 0.527 | 0.581 | 0.636 | 0.247 | 16.360 |
| | $\sigma_u^2$ | 2.753E-05 | 1.757E-06 | 2.428E-05 | 2.747E-05 | 3.112E-05 | -0.053 | 10.068 |
| | $\sigma_m^2$ | 7.082E-05 | 5.905E-06 | 6.016E-05 | 7.052E-05 | 8.319E-05 | -0.169 | 11.636 |
| | $\nu$ | 2.583 | 0.276 | 2.087 | 2.566 | 3.170 | -0.208 | 8.188 |
| June 5, 2018 | $c$ | 0.480 | 0.000 | 0.480 | 0.480 | 0.480 | NA | NA |
| model $\mathcal{T}$ | $a_1$ | 0.016 | 0.006 | 0.010 | 0.015 | 0.030 | 0.533 | 10.222 |
| | $a_2$ | 0.511 | 0.033 | 0.446 | 0.511 | 0.577 | -0.479 | 21.803 |
| | $\sigma_u^2$ | 1.052E-05 | 5.519E-07 | 9.490E-06 | 1.051E-05 | 1.165E-05 | -0.448 | 6.736 |
| | $\sigma_m^2$ | 5.838E-05 | 6.448E-06 | 4.708E-05 | 5.795E-05 | 7.232E-05 | 0.159 | 24.432 |
| | $\nu$ | 5.561 | 0.673 | 4.379 | 5.508 | 7.032 | 0.288 | 18.627 |
| June 6, 2018 | $c$ | 0.434 | 0.000 | 0.434 | 0.434 | 0.434 | NA | NA |
| model $\mathcal{T}$ | $a_1$ | 0.011 | 0.001 | 0.010 | 0.011 | 0.013 | 0.882 | 1.245 |
| | $a_2$ | 0.710 | 0.027 | 0.658 | 0.710 | 0.762 | -0.305 | 10.514 |
| | $\sigma_u^2$ | 1.360E-05 | 5.649E-07 | 1.253E-05 | 1.358E-05 | 1.475E-05 | 1.019 | 2.697 |
| | $\sigma_m^2$ | 2.572E-05 | 2.162E-06 | 2.192E-05 | 2.560E-05 | 3.030E-05 | 0.138 | 13.069 |
| | $\nu$ | 1.895 | 0.174 | 1.583 | 1.887 | 2.262 | -0.106 | 10.098 |
| June 7, 2018 | $\gamma$ | 19.521 | 0.383 | 18.547 | 19.604 | 19.981 | -0.346 | 4.716 |
| model $\mathcal{S}$ | $c$ | 0.395 | 0.003 | 0.388 | 0.395 | 0.401 | -1.414 | 5.009 |
| | $a_1$ | 0.013 | 0.003 | 0.010 | 0.012 | 0.021 | 0.084 | 3.151 |
| | $a_2$ | 0.632 | 0.032 | 0.571 | 0.632 | 0.695 | -1.254 | 26.244 |
| | $\sigma_u^2$ | 9.544E-06 | 5.322E-07 | 8.576E-06 | 9.524E-06 | 1.066E-05 | 1.989 | 7.170 |
| | $\sigma_m^2$ | 7.360E-05 | 6.985E-06 | 6.118E-05 | 7.314E-05 | 8.832E-05 | 0.733 | 24.932 |
| | $\nu$ | 7.733 | 0.831 | 6.240 | 7.688 | 9.513 | -0.019 | 15.930 |
| June 8, 2018 | $g$ | 0.297 | 0.042 | 0.223 | 0.296 | 0.387 | 1.397 | 71.989 |
| model $\mathcal{C}$ | $\sigma_u^2$ | 1.278E-03 | 1.488E-04 | 9.780E-04 | 1.281E-03 | 1.560E-03 | -1.355 | 56.898 |
| | $\sigma_m^2$ | 9.917E-03 | 1.446E-03 | 7.533E-03 | 9.733E-03 | 1.315E-02 | -1.544 | 56.839 |
| | $\nu$ | 7.793 | 0.984 | 6.048 | 7.736 | 9.859 | -0.926 | 16.567 |

In this table, the mean, standard deviation (Std.), 95% credible interval (CI95 lower and upper), median, z-value of convergence diagnostic test (CD) by Geweke (1992), and inefficiency factor (Inef.) by Chib (2001) for each sample and parameter are given.

below unity, which implies the adjustments were the under reaction to new information, especially that of June 8 was remarkably low.

## 6. Conclusion

This paper proposes a Bayesian approach to evaluate the intraday price discovery on a daily basis. Based on Amihud and Mendelson (1987), we introduce three candidate partial adjustment models: the models $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{C}$ whose adjustment coefficient is smoothly time-varying, switching abruptly, and being constant, respectively.

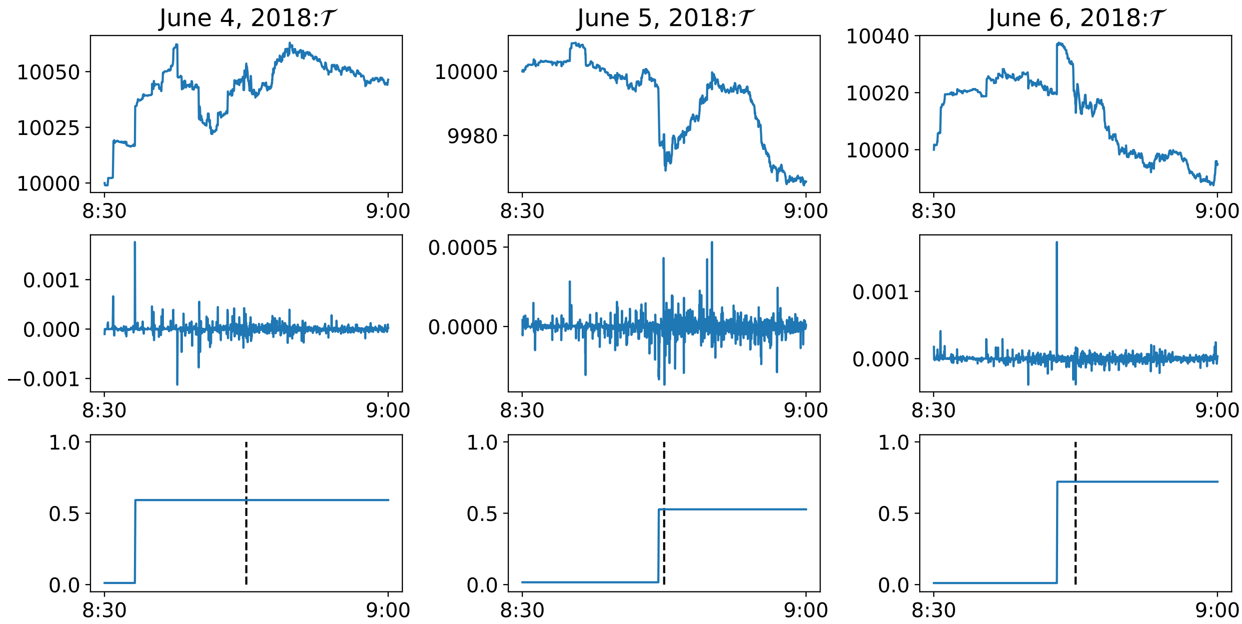


**Figure 4:** Price (top), return (middle) and the estimated path of adjustment coefficient (bottom) for June 4, 5 and 6, 2018, when the model $\mathcal{T}$ is selected.

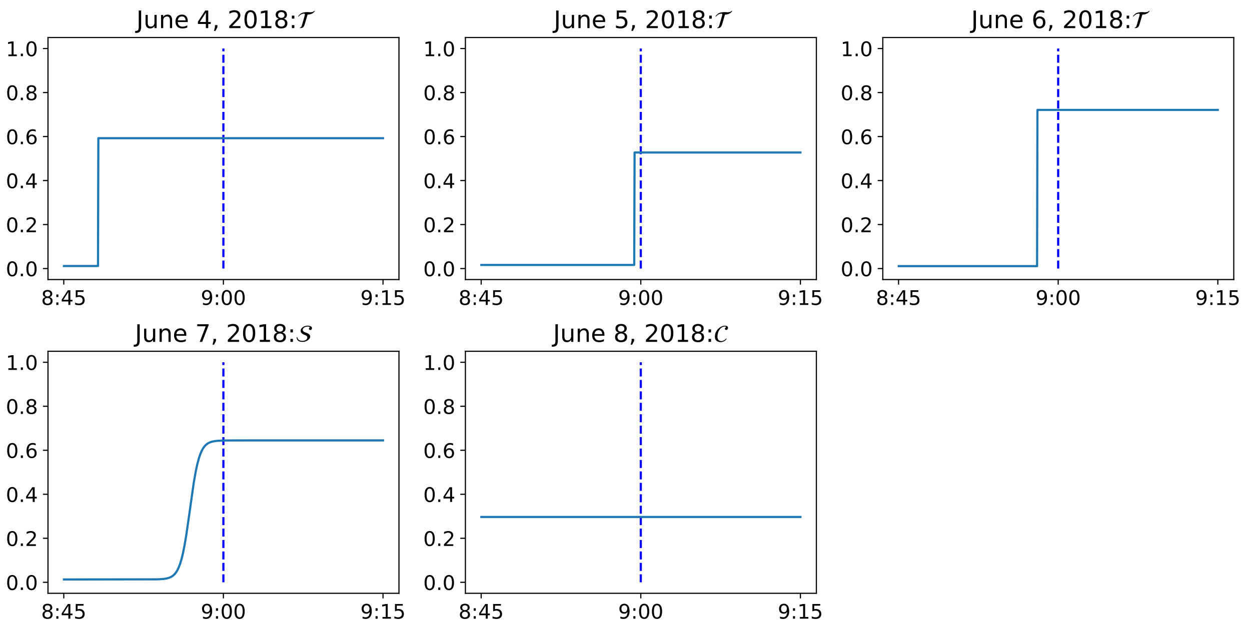


**Figure 5:** The estimation example of $\hat{s}_t$ for the real data from TOPIX Core 30 index

A simulation study shows that the proposed method aptly categorizes the type of price discovery. Though the model selection between model $\mathcal{S}$ and model $\mathcal{T}$ tends to be difficult for the case of less informative observed price data, the adjustment coefficient, that is the most important target, is correctly estimated even in such a case. The empirical study illustrates our methods using intraday high frequency time series of TOPIX Core30 of Tokyo Stock Exchange. We confirm that the proposed method gauge the price adjustment progress properly and find that the adjustment already starts before market opening via the indicative price information during preopening period.

We have considered the single transition scheme for the price adjustment in this paper. For the analysis of price discovery during the preopening period, the single transition is appropriate because the information that flows into the market is accumulated and reflected in the price as time passed whereas the analysis for the intraday period from open to close requires more flexible transition scheme. Future work could consider a model with a general path of progress of price discovery by considering three or more states.

## References


Amihud, Y. and H. Mendelson (1987): "Trading mechanisms and stock returns: an empirical investigation," *Journal of Finance*, 42, 533–553.

Biais, B., P. Hillion, and C. Spatt (1999): "Price discovery and learning during the preopening period in the Paris Bourse," *Journal of Political Economy*, 107, 1218–1248.

Carlin, B., A. Gelfand, and A. Smith (1992): "Hierarchical Bayesian analysis of changepoint problems," *Applied Statistics*, 41, 389–405.

Carter, C. and R. Kohn (1994): "On Gibbs sampling for state space models," *Biometrika*, 81, 541–553.

Chib, S. (1995): "Marginal likelihood from the Gibbs output," *Journal of the American Statistical Association*, 90, 1313–1321.

——— (2001): "Markov Chain Monte Carlo method: computation and inference," in *Handbook of Econometrics*, ed. by J. Heckman and E. Leamer, North Holland, Amsterdam, vol. 5, 3569–3649.

Chib, S. and I. Jeliazkov (2001): "Marginal likelihood from the Metropolis-Hastings output," *Journal of the American Statistical Association*, 96, 270–281.

Ekner, L. and E. Nejstgaard (2013): "Parameter identification in the logistic STAR model," *Discussion paper, Department of Economics, University of Copenhagen.*

Gerlach, R. and C. Chen (2008): "Bayesian inference and model comparison for asymmetric smooth transition heteroskedastic models," *Statistics and Computing*, 18, 391–408.

Geweke, J. (1992): "Evaluating the accuracy of sampling-based approaches to the calculation of posterior moments," in *Bayesian Statistics 4*, ed. by J. Bernardo, J. Berger, A. Dawid, and A. Smith, Oxford: Oxford University Press, 169–193.

Hasbrouck, J. (2007): *Empirical Market Microstructure*, Oxford University Press.

Kastner, G. and S. Frühwirth-Schnatter (2014): "Ancillarity-sufficiency interweaving strategy (ASIS) for boosting MCMC estimation of stochastic volatility models," *Computational Statistics and Data Analysis*, 76, 408–423.

So, M. and R. Chan (2014): "Bayesian analysis of tail asymmetry based on a threshold extreme value model," *Computational Statistics and Data Analysis*, 71, 568–587.

Spiegelhalter, D. J., N. G. Best, B. P. Carlin, and A. Van Der Linde (2002): "Bayesian measures of model complexity and fit," *Journal of the Royal Statistical Society. Series B: Statistical Methodology*, 64, 583–616.

Teräsvirta, T. (1994): "Specification, estimation, and evaluation of smooth transition autoregressive models," *Journal of the American Statistical Association*, 89, 208–218.

Tong, H. (1978): "On a threshold model," in *Pattern recognition and signal processing*, ed. by C. Chen, Amsterdam: Sijhoff and Noordhoff, 575–586.

Yu, Y. and X. L. Meng (2011): "To center or not to center: That is not the question-an ancillarity-sufficiency interweaving strategy (ASIS) for boosting MCMC efficiency," *Journal of Computational and Graphical Statistics*, 20, 531–570.