Naive Agents with Non-unitary Discounting Rate in a Monetary Economy

Koichi Futagami
Daiki Maeda

Discussion Paper 21-28

March 2022

Graduate School of Economics
Osaka University, Toyonaka, Osaka 560-0043, JAPAN
Naive Agents with Non-unitary Discounting Rate in a Monetary Economy*

Koichi Futagami† Daiki Maeda‡

February 25, 2022

Abstract

We incorporate naive agents with a non-unitary discounting rate into a cash-in-advance (CIA) model. Through this extension, we obtain the following results. First, we show that there exists an equilibrium in which the CIA constraint does not bind when individuals discount their utilities from future consumption lower than their utilities from future leisure time. It is important to note that this non-binding equilibrium exists even if the nominal interest rate takes a positive value. Second, we demonstrate that increases in the money supply growth rate decreases the individuals’ saving rate in the equilibrium in which the CIA constraint does not bind. Third, we exhibit that when the equilibrium where the CIA constraint does not bind exists, the welfare level of this equilibrium can be higher than that of the equilibrium in which the CIA constraint binds. Moreover, we deduce that the Friedman rule cannot be optimal in the equilibrium in which the CIA constraint binds and present the result that the optimal level of the optimal nominal interest rate is affected by the difference of the discount rates.

Keywords: Non-unitary discounting rate; Naive agents; CIA constraint; Monetary policy; Friedman rule

JEL classification: E52; E70

*Maeda is financially supported by the International Joint Research Promotion Program (Osaka University) and the Joint Usage/Research Center at the ISER.
†Department of Economics, Doshisha University, Karasuma-higashi-iru, Imadegawa-dori, Kamigyo-ku, Kyoto 602-8580, Japan, E-mail: k1godgod@gmail.com
‡School of Global Studies, Chukyo University, 101-2 Yagoto-Honmachi, showa-ku, Nagoya, Aichi, 466-8666, Japan. E-mail: d-maeda@mecl.chukyo-u.ac.jp
1 Introduction

Cash-in-advance (CIA) models have played important roles in monetary economics. They assume that money stock must be held before purchases of goods. We start our analyses on individuals’ behavior in the monetary economy and on policies in such an economy using a CIA model. However, we incorporate new developments of behavioral economics into our analyses. We suppose that individuals derive their utility from consumption of goods and leisure time. It is further assumed that they discount their utility from consumption and leisure time at different rates\(^1\). Hori and Futagami (2019) propose this non-unitary discount rate (NUDR) model and show that this discounting behavior makes individuals’ decision time inconsistent\(^2\). That is, a future decision made by an individual in the present may change; the individual’s present decision may differ from that made in the past\(^3\). Thus, we will show that taking account of this behavior can change results obtained based on standard CIA models in which individuals discount their utility from consumption and leisure time at the same rates. We call this case the unitary discount rate (UDR) model. Specifically, we pose the following issues: How do such individuals behave under the CIA constraint? How should money supply be determined in such an environment? Does the Friedman rule hold in such an economy?

The above setting requires that we must treat an individual at different times as different individuals\(^4\). We refer to them as selves. Consequently, we must consider a game theoretical situation among selves and seek a solution to such an intra-personnel game. So far, most preceding studies have focused on the case where individuals recognize that their preferences change over time, exhibiting that individuals are sophisticated. These studies solve Nash equilibria among selves; however, individuals may not be so sophisticated in the real world. In contrast, the present study considers that agents do not recognize that they change their preferences in the future, that is, agents are naive. This setting makes

\(^1\)Fredrick et al. (2002) point out that individuals may apply different rates when discounting their utility from different sources.
\(^2\)Ubfal (2016) shows empirical evidence for this discounting way. The other researches based on empirical evidence are given in Hori and Futagami (2019) and Hori, Futagami, and Morimoto (2021).
\(^3\)Hyperbolic discounting also induces the time-inconsistency problem to individual’s decision. See Laibson (1996, 1997, 1998).
\(^4\)See for example Peleg and Yaari (1973) and Goldman (1980).
individuals’ behavior time-inconsistent and yields unique results, as explained below.

We use a simple CIA model with capital accumulation in which the CIA constraint is applied only to purchases of consumption goods. The main results are as follows: First, we find that there exists not only the equilibrium in which the CIA constraint binds but also one in which the CIA constraint does not bind, when individuals discount their utilities from consumption lower than their utilities from future leisure time. It is important to note that this non-binding equilibrium exists even if the nominal interest rate takes a positive value. This implies that individuals would like to hold more money for future consumption rather than spending present consumption, even if holding money is costly; that is, money plays a role in the transmission of wealth over periods in the present NUDR model with the non-binding CIA constraint.

Second, the study reveals that increases in the money supply growth rate do not affect the individuals’ saving rate in the equilibrium where the CIA constraint binds. However, increases in the growth rate of money supply decreases the individuals’ saving rate in the equilibrium in which the CIA constraint does not bind. Moreover, the latter result is in sharp contrast with the UDR model’s well-known result; which states when the CIA constraint is imposed only on purchases of consumption goods in the UDR model, the individuals’ saving behavior is independent of the money supply rate

Third, we conduct welfare analyses using a welfare measure from the standpoint of the initial self, as in Krusell et al. (2002). Thus, the study shows that when the equilibrium where the CIA constraint does not bind exists, the welfare level of this equilibrium can be higher than that of the equilibrium in which the CIA constraint binds. Moreover, we analyze the optimal monetary policy. Then, we analytically show that the Friedman rule cannot be optimal in the equilibrium in which the CIA constraint binds and present the result that the optimal level of the nominal interest rate is affected by the difference of the discount rates using the numerical analysis.

---

[5] If there are CIA constraints not only on consumption but also on investment, the growth rate of money affects the saving rate. This is shown by Stockman (1981) and Abel (1985). However, in our model, even if there is no CIA constraint on investment, the growth rate of money affects the saving rate.

[6] Using a quasi-geometric discounting model that induces the time-inconsistency problem, Krusell et al. (2002) compared the allocation of market economy with that of the central planner, the benevolent government. They conclude that the market economy can do its job better than the central planner.
The most related study to the present one is Hori, Futagami, and Morimoto (2021). They also use a NUDR model without capital and show that the Friedman rule is not optimal when individuals discount utilities from consumption higher than future leisure time. In contrast with our model, individuals in their model are sophisticated. In addition, their model does not have capital stock. Thus, they do not explore the relationship between money supply and saving behavior. Other literature based on hyperbolic discounting that focuses on the monetary economy are as follows: Maeda (2018) uses a similar CIA model with capital accumulation to ours in which the CIA constraint is applied only to purchases of consumption goods except for the discounting way. He shows that individuals engage in over-saving and the optimal growth rate of money supply is lower when individuals use quasi-geometric discounting than when individuals use usual geometric discounting. Graham and Snower (2013) show that the Friedman rule fails when individuals use hyperbolic discounting based on a New Keynesian model. Their focus is on the interplay between hyperbolic discounting and nominal wage rigidity. In contrast with their study, we consider the standard neoclassical growth model with flexible prices. Boulware, Reed, and Ume (2013) show that the welfare cost of inflation is higher if individuals are myopic. In contrast with our model, they use money in the utility model without capital. A different approach from the hyperbolic discounting framework is from Hiraguchi (2018). He uses a search model where individuals have Gul-Pesendorfer (2001) preferences and shows that the Friedman rule is not optimal. However, although temptation goods exist in his model, the temptation goods do not cause time inconsistency.

The remainder of this paper is organized as follows: Section 2 provides the model. Section 3 explains the agent’s problem. Section 4 obtains the equilibrium. Section 5 presents an analysis of welfare and government policy. Section 6 concludes the paper.
2 The model

Time is discrete and denoted \( t = 0, 1, 2, \cdots \). In this economy, a final good is produced. The production function of the good is given by

\[
y_t = A \bar{k}_{t-1}^{1-\alpha} \bar{l}_{t},
\]

where \( y_t \) denotes the amount of production of final goods, \( \bar{k}_{t-1} \) and \( \bar{l}_t \) denote aggregated capital and labor, respectively, \( A \) denotes productivity and \( \alpha \in (0, 1) \) denotes capital share. We assume that the depletion rate of capital is one. Then, the market clearing condition of the final goods is given by

\[
A \bar{k}_{t-1}^{\alpha} \bar{l}_{t}^{1-\alpha} = \bar{c}_t + \bar{k}_t,
\]

where \( \bar{c}_t \) is the aggregated consumption. Moreover, we assume that the final goods market is perfectly competitive.

We also assume that the factor markets are also perfectly competitive. Then, the factor prices are equal to each marginal product as follows:

\[
r_t = \alpha A \bar{k}_{t-1}^{\alpha-1} \bar{l}_{t}^{-\alpha},
\]

\[
w_t = (1 - \alpha) A \bar{k}_{t-1}^{\alpha} \bar{l}_{t}^{-\alpha},
\]

where \( r_t \) and \( w_t \) denote the real gross interest rate and real wage, respectively.

In this economy, a government issues money. We assume that the growth rate of nominal money is \( \mu \) and constant. Then, the dynamics of money is given by

\[
\bar{m}_t = \frac{1 + \mu}{1 + \pi_t} \bar{m}_{t-1},
\]

where \( \bar{m}_t \) denotes a real money balance at the end of period \( t \) and \( \pi_t \) is an inflation rate. The government transfers all issued money to households at the beginning of each period.
Then, the real transfer is given by

$$\tau_t = \frac{\mu_m \tau_{t-1}}{1 + \pi_t}.$$  \hspace{1cm} (6)

In this economy, there is a continuum of infinitely lived homogeneous agents with a unit measure. Each agent has a unit of time and divides it into leisure and labor supply. The agent has the following lifetime utility function:

$$U_t = \sum_{i=0}^{\infty} \beta_c^i u_c(c_{t+i}) + \beta_l^i v_l(1 - l_{t+i}),$$  \hspace{1cm} (7)

where $c_t$ and $l_t$ denote the amount of consumption and labor supply, $u_c$ and $v_l$ are instantaneous utility functions of consumption and leisure. $\beta_c$ and $\beta_l \in (0, 1)$ denote discount factors of consumption and leisure, respectively. We call the case where $\beta_c \neq \beta_l$ as “non-unitary discounting” and the case where $\beta_c = \beta_l$ as “unitary discounting”. In this study, we assume that the instantaneous utility functions are given by

$$u_c(c_t) = \ln c_t,$$  \hspace{1cm} (8)

$$v_l(1 - l_t) = \theta \ln(1 - l_t).$$  \hspace{1cm} (9)

The agent’s preference changes over time when the discount rates differ; (see Hori and Futagami [2019]). Therefore, it is important whether the agent understands this change. In this study, we assume that all agents do not understand this change. In other words, they are naive. There are two assets in this economy, capital and money. Therefore, the real budget constraint is given by

$$k_t + m_t = r_t k_{t-1} + w_t l_t + \frac{m_{t-1}}{1 + \pi_t} + \tau_t - c_t,$$  \hspace{1cm} (10)

where $k_t$ and $m_t$ denote the agent’s capital and real money holdings at the end of period $t$. The agent demands money because he/she faces the cash-in-advance (CIA) constraint
on consumption as follows:

\[ c_t \leq \frac{m_{t-1}}{1 + \pi_t} + \tau_t. \tag{11} \]

### 3 The agent’s problem

In this section, we define and solve the agent’s problem. We first give some explanations of notations. Hereafter, for simplification, we omit the time subscript “\(t\)”. Moreover, the stock variables omitted subscript denote variables at the beginning of the period, and variables added prime denote those at the end of the period. Taking the final goods market clearing condition (2) as an example, we can express it as \(A\bar{k}^\alpha \bar{l}^{1-\alpha} = \bar{c} + \bar{k}'\).

First, we illustrate the agent’s current behavior. The current value function at the initial time is given by

\[
V_0(k, m, \bar{k}, \bar{m}) = \max_{c, k', \bar{k}', \bar{m}'} \left[ \ln c + \theta \ln (1 - l) + V(k', m', \bar{k}', \bar{m}') \right],
\tag{12}
\]

where \(\ln c + \theta \ln (1 - l)\) is the utility perceived in the current period, and \(V(k', m', \bar{k}', \bar{m}')\) is the future value function predicted by the current agent. The constraints for the current agent are the budget constraint (10) and the CIA constraint (11). The agent’s current value function depends not only on his/her stock variables \((k \text{ and } m)\) but also on the aggregate stock variables \((\bar{k} \text{ and } \bar{m})\) because these determine the factor prices. Then, we obtain the following first order conditions:

\[
\frac{1}{c} - \lambda - \lambda_{CIA} = 0, \tag{13}
\]

\[
-\frac{\theta}{1 - l} + w\lambda = 0, \tag{14}
\]

\[
-\lambda + \frac{\partial V(k', m', \bar{k}', \bar{m}')}{{\partial k'}} = 0, \tag{15}
\]

\[
-\lambda + \frac{\partial V(k', m', \bar{k}', \bar{m}')}{{\partial m'}} = 0, \tag{16}
\]

where \(\lambda\) and \(\lambda_{CIA}\) are the Lagrange multipliers associated with the budget constraint and the CIA constraint, respectively. The future value function is explained below.
The naive agent does not understand that his/her preference changes over time. Therefore, the current agent considers that the future self also maximizes (12). As the future value is equal to the discounted sum of the future utilities, it is given by

$$V(k, m, \bar{k}, \bar{m}) = \max_{c_p, l_p, k_p, m_p} \left[ \beta_c V_c(k, m, \bar{k}, \bar{m}) + \beta_l V_l(k, m, \bar{k}, \bar{m}) \right],$$

where

$$V_c(k, m, \bar{k}, \bar{m}) = \sum_{i=0}^{\infty} \beta_c^i \ln c_{p, t+i} = \ln c_p + \beta_c V_c(k'_p, m'_p, \bar{k}'_p, \bar{m}'_p),$$

$$V_l(k, m, \bar{k}, \bar{m}) = \sum_{i=0}^{\infty} \beta_l^i \ln(1 - l_{p, t+i}) = \theta \ln(1 - l_p) + \beta_l V_l(k'_p, m'_p, \bar{k}'_p, \bar{m}'_p).$$

The variables added the subscript p denote the predicted variables. The future constraints are same as those of the current problem, (12). Then, the first order conditions of (17) are given by

$$\frac{\beta_c}{c_p} - \lambda_p - \lambda_{p, CIA} = 0,$$

$$- \frac{\beta_l \theta}{1 - l_p} + w_p \lambda_p = 0,$$

$$-\lambda_p + \frac{\partial^2 V_c(k'_p, m'_p, \bar{k}'_p, \bar{m}'_p)}{\partial k'_p} + \frac{\partial^2 V_l(k'_p, m'_p, \bar{k}'_p, \bar{m}'_p)}{\partial k'_p} = 0,$$

$$-\lambda_p + \frac{\partial^2 V_c(k'_p, m'_p, \bar{k}'_p, \bar{m}'_p)}{\partial m'_p} + \frac{\partial^2 V_l(k'_p, m'_p, \bar{k}'_p, \bar{m}'_p)}{\partial m'_p} = 0.$$
given by $\tilde{k}'_p = k'_p$, $\tilde{l}_p = l_p$ and $\tilde{m}'_p = m'_p$. The factor prices are determined based on these, and then all the agents can predict their future selves’ behavior.

We also assume the following:

**Assumption 2.** The agents predict that the CIA constraint will be binding in the future.

From the preceding arguments, we can summarize the agents’ predictions as the following definition:

**Definition 1.** The naive agent’s prediction consists of the quantities $\{k'_p, l_p, m'_p, \tilde{k}'_p, \tilde{l}_p, \tilde{m}'_p\}$ and the prices $\{r_p, w_p, \pi_p\}$, which satisfy the following conditions.

1. The agents behave the money supply rule, $\mu$, as given.

2. The agents’ optimal conditions are given by equations (20) to (23).

3. The agents predict that other agents behave as Assumption 1 states.

4. The predicted CIA constraint is binding as Assumption 2 states.

5. The firms’ predicted profit maximizing conditions are given by (3) and (4).

6. The predicted conditions satisfy: $k'_p = \tilde{k}'_p$, $l_p = \tilde{l}_p$, $m'_p = \tilde{m}'_p$ and (2).

## 4 Equilibrium

In this section, we first define equilibria of our NUDR model with naive agents and then show that there can be two types of equilibria\(^7\). The first one is the case where the “current” CIA constraint is binding, and the second one is the case where it is not binding.

\(^7\)The definitions of the equilibria discussed in this section correspond to “a temporary competitive equilibrium” presented by Gabrieli and Ghosal (2013).
4.1 The equilibrium in which the current CIA constraint is binding

We can define the first equilibrium in which the CIA constraint is binding as follows.

**Definition 2.** The equilibrium in which the CIA constraint is binding consists of the quantities \( \{k', l, m', \bar{k}', \bar{l}, \bar{m}'\} \) and the prices \( \{r, w, \pi\} \), which satisfy the following conditions.

1. The agents behave the money supply rate, \( \mu \), as given.

2. The agents’ predictions satisfy Definition 1.

3. The agents’ optimal conditions are given by equations (13) to (16).

4. The CIA constraint is binding.

5. The firms’ profit maximizing conditions are given by (3) and (4).

6. The equilibrium becomes symmetric: \( k' = \bar{k}' \), \( l = \bar{l} \) and \( m' = \bar{m}' \).

7. The market clearing condition (2) is satisfied.

Hereafter, we call this equilibrium as the **Binding Equilibrium (BE)**. Using the guess and verify method, we obtain the following proposition.

**Proposition 1.** In the BE, the saving rate and the labor supply are respectively given by

\[
\begin{align*}
    s^* &= \alpha \beta_c - \frac{\beta_1^2}{1-\beta_c} + \theta \frac{\beta_1}{1-\beta_1} - \frac{\alpha \theta_2}{1-\beta_1} (\beta_1 - \beta_c), \\
    l^* &= \frac{1 - \alpha}{1 - \alpha + \theta (1 - s^*) \frac{1+\mu}{\beta_c}}.
\end{align*}
\]

Moreover, the real money balance and the inflation rate are respectively given by

\[
\begin{align*}
    \bar{m}^* &= (1 - s^*) A k^a (l^*)^{1-\alpha}, \\
    \pi^* &= \frac{(1 + \mu) \bar{m}}{(1 - s^*) A k^a (l^*)^{1-\alpha}} - 1.
\end{align*}
\]
However, because we assume that the predicted CIA is binding, the following inequality must be satisfied:

\[
1 + \mu \geq 1 + \mu = \frac{\beta_l^2}{1-\beta_l} + \theta \frac{\beta_l^2}{1-\beta_l}.
\]  

(28)

Proof. See Appendix A.

Based on this proposition, we find that this equilibrium has two properties. The first one is that the growth rate of money does not affect the saving rate and only affects the labor supply. This property is the same as the case of unitary discounting. The second one can be stated as the following corollary:

**Corollary 1.** If the discount factor of the utility from leisure, \(\beta_l\), is higher (lower) than that of the utility from consumption, \(\beta_c\), then the saving rate is higher (lower) than the case where discount factors are the same between those.

Proof. When \(\beta_l = \beta_c\), \(s^*|_{\beta_l=\beta_c} = \alpha \beta_c\). If \(\beta_l > (<) \beta_c\), we obtain

\[
\frac{\beta_l^2}{1-\beta_l} + \theta \frac{\beta_l^2}{1-\beta_l} - \frac{\alpha \theta \beta_l^2}{1-\beta_l} (\beta_l - \beta_c) > (<) 1.
\]  

(29)

Therefore, we obtain this corollary.

The reason we obtain this corollary is the following. If the discount factor of consumption is relatively low (high), then the agents want to hold less (more) money for the less (much) future consumption. As a result, the agents invest more (less) in capital.

### 4.2 The equilibrium in which the current CIA constraint is not binding

We can also define the equilibrium when the current CIA constraint is not binding. Hereafter, we call this equilibrium the *not binding equilibrium* (NBE). We can show that there
exists an equilibrium in which the current CIA constraint is not binding. Then, we obtain the following proposition for the saving rate and the labor supply.

**Proposition 2.** There is the NBE if the following inequalities hold:

\[ 1 + \mu \leq 1 + \mu < \beta_c, \]  

where

\[ 1 + \mu = \frac{\beta_1^{2}}{1 - \beta_1} + \theta \frac{\beta_1^{2}}{1 - \beta_1}. \]

In the NBE, the saving rate, labor supply, and real money balance are given by

\[ s^{**} = \frac{\beta_1^{2}}{1 - \beta_1} + \theta \frac{\beta_1^{2}}{1 - \beta_1}, \]

\[ l^{**} = \frac{1 - \alpha}{1 - \alpha + \theta(1 - s^{**})}, \]

\[ m^{**} = \frac{\beta_c(1 + F)}{(1 + F + \beta_1^{2}/1 - \beta_1 + \theta \beta_1^{2}/1 - \beta_1)(1 + \mu)} - A\kappa \tilde{l}^{1 - \alpha}, \]

where

\[ 1 + F \equiv \frac{\beta_1^{2}}{1 - \beta_1} + \theta \frac{\beta_1^{2}}{1 - \beta_1} - \alpha \left( \frac{\beta_1^{2}}{1 - \beta_1} + \theta \frac{\beta_1^{2}}{1 - \beta_1} \right), \]

\[ m^{**} = \frac{\beta_c(1 + F)}{(1 + F + \beta_1^{2}/1 - \beta_1 + \theta \beta_1^{2}/1 - \beta_1)(1 + \mu)} - A\kappa \tilde{l}^{1 - \alpha} - 1. \]

**Proof.** See Appendix B.

Combining (5) and (34), we can obtain the inflation rate as follows:

\[ \pi^{**} = \alpha \beta_c \left( 1 + F + \frac{\beta_1^{2}}{1 - \beta_1} + \theta \frac{\beta_1^{2}}{1 - \beta_1} \right)(1 + \mu) \frac{\tilde{m}}{A\kappa (l^{**})^{1 - \alpha} - 1}. \]

As \( 1 + F \) is an increasing function of \( \mu \) from (35) and \( l^{**} \) is a decreasing function of \( \mu \), the inflation rate increases with \( \mu \). Therefore, inequality (30) means that the low inflation rate causes the NBE. The low inflation rate means that the cost of holding money is low.
Therefore, the agents can hold sufficient money to consume more in the future and thus the NBE exists.

In addition, we can state the following corollary that shows when condition (30) holds.

**Corollary 2.** If the discount factor of consumption is higher than that of leisure, that is \( \beta_c > \beta_l \), there exists \( \mu \) which satisfies condition (30).

**Proof.** Using (31), we obtain

\[
1 + \mu - \beta_c = \frac{\beta_l}{1 - \beta_l} (\beta_l - \beta_c) + \frac{\beta_l^2}{1 - \beta_l} + \theta \frac{\beta_l}{1 - \beta_l}.
\]

From the above equation, if \( \beta_c > \beta_l \), we obtain the following inequality: \( 1 + \mu < \beta_c \). This implies that there exists \( \mu \) which satisfies (30). In other words, the NBE can exist. \( \square \)

The higher discount factor of consumption implies that the marginal utility of future consumption is high. Then, the agent holds sufficient money to consume more in the future. Therefore, if \( \beta_c > \beta_l \) is satisfied, the NBE exists.

Next we characterize the saving rate \( s^* \) of the NBE. From (32), we obtain the following proposition.

**Proposition 3.** An increase in the growth rate of money supply, \( \mu \), decreases the saving rate. Moreover, if \( 1 + \mu \) approaches \( \beta_c \), \( s^* \) approaches \( s^* \).

**Proof.** From (35), \( \frac{\partial s^*}{\partial \mu} 1 + F > 0 \). Therefore, \( \frac{\partial s^*}{\partial \mu} < 0 \) from (32). Moreover, substituting \( \beta_c \) into \( 1 + \mu \) in (32), we obtain \( s^* \). \( \square \)

From (A.39) in Appendix A, an increase in \( \mu \) raises the future predicted inflation rate. Then, the agents have an incentive to decrease money holdings because higher inflation rates mean higher costs of holding money. The agents cannot increase their consumption if the CIA constraint is binding. However, they can increase the current consumption in the NBE. Therefore, in the NBE, the agents can increase the current consumption and decrease the money holdings. From the final goods market clearing condition, investment in capital decreases if the current consumption increases. This is why the saving rate is a decreasing function of the growth rate of money supply \( \mu \). In the UDR models with
the CIA constraint on consumption, the growth rate of money supply does not affect the saving rate. However, in our model, it does. This result in our NUDR model is quite different from that of the UDR models.

In addition, we obtain the following proposition:

**Proposition 4.** The NBE exists in the steady state even if the nominal interest rate is positive.

*Proof.* We consider the nominal interest rate in the steady state. We add a tilde to the variables in the steady state. The inflation rate is given by \( \tilde{\pi} = \mu \). As \( \tilde{k}' = \tilde{k} \) in the steady state, we obtain \( \alpha A \tilde{k}^{\alpha-1} (\tilde{r}^{**})^{1-\alpha} = \alpha / s^{**} \) from (B.1) in Appendix B. This is the real gross interest rate (\( \tilde{r} \)). Therefore, the nominal interest rate is given by

\[
\tilde{i} = (1 + \tilde{\pi})(1 + \tilde{r}) - 1 = \frac{1 + \mu}{s^{**}} - 1. \tag{37}
\]

From Proposition 3, the saving rate in the NBE approaches \( s^* \) when \( \mu \) approaches the upper bound. This upper bound of \( \mu \) is given by \( \beta_c - 1 \) from (30). That is, \( s^{**} \) approaches \( s^* \) if \( \mu \) approaches \( \beta_c - 1 \). Then, the nominal interest rate is given by

\[
\lim_{\mu \to \beta_c - 1} \tilde{i} = \alpha \frac{\beta_c}{s^*} - 1 = -\frac{\alpha \beta_c}{1 - \beta_c} \frac{\beta_1 - \beta_c}{\delta^2} + \theta \frac{\beta_1}{1 - \beta_c} \tag{38}
\]

From Corollary 2, we find that \( \lim_{\mu \to \beta_c - 1} \tilde{i} > 0 \) in the NBE. Therefore, we have shown that the NBE exists even if the nominal interest rate is positive.

In standard models, individuals do not have any incentive to hold money that is not used to purchase consumption goods when the nominal interest rate takes positive values, which means that holding money is costly. In our model, the individuals predict that their future self uses all holding money; that is, the individuals hold money to trade. Because of time-inconsistent preference, the current selves change their behavior based on predictions and does not use all holding money even when the future arrives.
5 Welfare and policy

This section discusses optimal monetary policy. The first subsection defines a welfare function and derives the optimal saving rate and labor supply. In the following, we examine optimal monetary policy in the BE and the NBE, respectively.

5.1 The optimal saving rate and labor supply

Following Krusell et al. (2002), we adopt the lifetime utility of the initial self, (7) as the welfare measure. If the saving rate and labor supply are constant as in the equilibrium in this model, the welfare level is given by

\[
W = \frac{1}{1 - \beta_c} \ln(1 - s) + \frac{1}{(1 - \beta_c)(1 - \alpha \beta_c)} \ln A + \frac{\alpha \beta_c}{(1 - \beta_c)(1 - \alpha \beta_c)} \ln s + \frac{\alpha}{1 - \alpha \beta_c} \ln k
\]
\[
+ \frac{1 - \alpha}{(1 - \beta_c)(1 - \alpha \beta_c)} \ln l + \frac{\theta}{1 - \beta_l} \ln(1 - l). 
\]

The saving rate and the labor supply that maximize (39) are

\[
s_{op} = \alpha \beta_c, 
\]
\[
c_{op} = \frac{1 - \alpha}{1 - \alpha + \theta(1 - s_{op})^{1 - \beta_c}}. 
\]

5.2 The welfare and the optimal policy in the BE

Comparing the saving rate and labor supply in the market equilibrium with the optimal ones, we obtain the following proposition.

**Proposition 5.** If the discount factor of consumption, $\beta_c$, is lower (higher) than that of leisure, $\beta_l$, the saving rate in the BE is higher (lower) than the optimal one.

**Proof.** When these discount factors are equal, the corresponding saving rate is equal to the optimal one. The proposition obtains from Proposition 1, (40), and (41). □

As the saving rate in the equilibrium does not depend on the growth rate of money, the optimal allocation for the initial self cannot be attained by using the growth rate
of money. Thus, we consider only the labor supply to obtain the second-best monetary policy. We obtain the following lemma.

**Lemma 1.** The second-best monetary policy is given by

$$1 + \mu^{SB} = \beta_c \frac{1 - \beta_c}{1 - \beta_l} \frac{1 - s^{op}}{1 - s^*},$$

where $\mu^{SB} \geq \underline{\mu}$. Otherwise, $\mu^{SB} = \underline{\mu}$.

**Proof.** The labor supply coincides with the optimal one for the initial self when it is equal to $l^{op}$. Therefore, the second-best $\mu$ is determined to satisfy $l^* = l^{op}$. This equality leads to (42).

If $\mu^{SB} < \underline{\mu}$, the lower bound of labor supply is lower than the optimal one. In this case, the labor supply cannot be increased by decreasing $\mu$ although welfare improves if so. Therefore, the second-best policy is $\underline{\mu}$ in this case.

Comparing the second-best policy with the Friedman (1969) rule, we obtain the following proposition.

**Proposition 6.** The second-best monetary policy deviates from the Friedman rule if

$$\frac{1 - \beta_c (1 - s^{op})s^{op}}{1 - \beta_l (1 - s^*)s^*} > 1.$$ 

**Proof.** Similar to the proof of Proposition 4, we can obtain the inflation rate in the steady state as follows:

$$\tilde{i} = \alpha \left( \frac{1 + \mu}{s^*} \right) - 1.$$

The second-best monetary policy deviates from the Friedman rule if the optimal nominal interest rate takes a positive value in the steady state. Substituting (42) into (44), we
obtain

\[
\tilde{i}^{SB} = \frac{1 - \beta_c (1 - s^{op}) \alpha \beta_c}{1 - \beta_l (1 - s^*) s^*} - 1.
\]

As \( s^{op} = \alpha \beta_c \), we have shown that \( \tilde{i}^{SB} > 0 \) when the inequality (43) holds.

We conduct numerical analyses to examine the effects of the relationship between the discount factors and the Friedman rule. We set \( \alpha \) to 0.3 because the labor share in Japan is about 0.7^{8}. We assume that \( \beta_c \) coincides standard macroeconomic model. Following Cooley and Prescott (1995), we set \( \beta_c \) to 0.95, which is an annual discount factor. \( \theta \) is set to 0.75. Figure 1 shows the result. The horizontal axis is \( \beta_l \). The solid line shows the left-hand side and the dashed line shows the right-hand side of (43). From the figure, we find that the second-best policy deviates from the Friedman rule when the discount factor of leisure, \( \beta_l \), is higher than that of consumption, \( \beta_c \). This result is robust even if \( \theta \) changes. From (40) and (41), the optimal labor supply is a decreasing function of \( \beta_l \). This means that if \( \beta_l \) is high, the optimal labor supply is low and it becomes lower than the equilibrium one. The equilibrium labor supply decreases when \( \mu \) increases. Therefore,

---

^{8}See Figure 1 in Fukao and Perugini (2021).
under the above parameter setting, the high nominal interest rate caused by the increase in \(\mu\) becomes better for the initial self. Thus the second best monetary policy deviates from the Friedman rule\(^9\).

### 5.3 The welfare and the optimal policy in the NBE

In the NBE, we can state the following proposition:

**Proposition 7.** If \(\beta_c > \beta_l\) and \(\mu\) is sufficiently close to \(\beta_c\), the welfare level of the initial self in the NBE is higher than that of the BE.

**Proof.** From Proposition 5, \(s^\text{op} > s^*\) if \(\beta_c > \beta_l\). From Proposition 3, \(s^{**} > s^*\) and \(s^{**}\) goes to \(s^*\) when \(1 + \mu\) approaches to \(\beta_c\). Thus, there exists \(s^{**}\) which satisfies \(s^\text{op} > s^{**} > s^*\).

Comparing (25) and (41), we obtain \(l^\text{op} > l^*\) because \(\beta_c > \beta_l\) and \(\beta_c > 1 + \mu\) in the NBE. From (25) and (34), \(l^{**} > l^*\) because \(\beta_c > 1 + \mu\). As \(s^{**}\) goes to \(s^*\), \(l^{**}\) also goes to \(l^*\). Thus, there exists \(l^{**}\) which satisfies \(l^\text{op} > l^{**} > l^*\). The second order partial derivatives of the welfare level with respect to \(s\) and \(l\) are respectively given by

\[
\frac{\partial^2 \mathcal{W}}{\partial s^2} = \frac{1}{(1 - \beta_c)(1 - s)^2} - \frac{\alpha \beta_c}{(1 - \beta_c)(1 - \alpha \beta_c)s^2} < 0,
\]

\[
\frac{\partial^2 \mathcal{W}}{\partial l^2} = \frac{1 - \alpha}{(1 - \beta_c)(1 - \alpha \beta_c)l^2} - \frac{\theta}{(1 - \beta_l)(1 - l)^2} < 0.
\]

Therefore, we obtain this proposition. \(\square\)

From (40) and (41), both optimal saving rate and labor supply becomes high if \(\beta_c\) is high. This is why the value of future consumption is high. Therefore, the saving rate in the NBE can become lower than the optimal one. As shown in the proof of Proposition 7, the saving rate in the NBE is higher than that in the BE. Therefore, the welfare level in the NBE is higher than that of the BE. From Proposition 2, the NBE exists if the government induces the growth rate of money to be \(1 + \mu < \beta_c\). This implies that if the government increases the growth rate of money and the inflation rate\(^10\), the welfare improves.

\(^9\)We obtain the same result when we set \(\alpha\) to 0.4, which is the capital share of the US. The data source is Feenstra et al. (2015).

\(^{10}\)See equation (36).
The curved surface shows welfare in the NBE. The red area is an area in which $\underline{\mu} \leq \mu$. The parameters $\alpha$, $\beta_c$, and $\theta$ are set 0.3, 0.95, and 0.75, respectively.

We next discuss the monetary policy in the NBE. From (32), (33), (40), and (41), $s^{**} = s^{op}$ and $\bar{l}^{**} = \bar{l}^{op}$ cannot be simultaneously satisfied because the government can use only one policy instrument, that is, the growth rate of money supply. Therefore, we consider the second-best policy in the NBE. We first seek the second-best policy from the numerical example. Figure 2 shows the welfare of the NBE. In the red area of this figure, the condition in which there exists the NBE, that is $\underline{\mu} \leq \mu$ is satisfied. The parameters are the same as those of subsection 5.2. We find that welfare is maximized when $\underline{\mu} = \mu$ in the feasible growth rate of money. Therefore, the second-best policy is $\underline{\mu}$ in this numerical example.

We next numerically discuss whether the second-best monetary policy satisfies the Friedman rule. Figure 3 shows the nominal interest rate in the steady state when $\mu = \underline{\mu}$. From this figure, we find that the nominal interest rate under the second-best monetary policy is negative\textsuperscript{11}. Although the nominal interest rate is lower than zero, the second-

\textsuperscript{11}The money market is clearing although the nominal interest rate is negative. This is because $\underline{\mu}$ is the growth rate of money which induces the predicted future inflation rate to be zero. See page 26 to 27.
best monetary policy is qualitatively the same as the Friedman rule. The Friedman rule is that the inflation rate is induced to the lower bound. Under this numerical example, the second-best monetary policy is also induced to the lower bound. From this result, the government should induce the nominal interest rate as low as possible when \( \beta_c > \beta_l \) in which the NBE occurs.

Combining the two numerical examples in this subsection and subsection 5.2, the optimal steady state inflation rate (nominal interest rate) is high if the discount factor of leisure is higher than that of consumption, and it is low if vice versa. This implies that the difference between the discount rates of consumption and leisure affects the optimal monetary policy, and thus the government should consider the difference in discount rates.

6 Conclusion

We have developed the monetary model with non-unitary discounting and naive agents who face a cash-in-advance constraint on purchases of consumption goods. Thus, we have obtained the following characteristic results.

First, there is an equilibrium in which the current CIA constraint is not binding in Appendix A for the details.
even if the nominal interest rate is positive. Second, increases in the growth rate of money supply decreases the individuals' saving rate in the equilibrium in which the CIA does not bind. These two results are quite different from standard models with the same setting except for the assumptions of non-unitary discounting and naive agents. Third, the welfare in the equilibrium in which the CIA constraint is not binding can be higher than that in the equilibrium in which the CIA constraint is binding if the discount factor of consumption is higher than that of leisure. Fourth, the difference between the discount factors of consumption and leisure affects the optimal monetary policy. The optimal nominal interest rate is high if the discount factor of leisure is higher than that of consumption, and it is low if the discount factor of leisure is lower than that of consumption. This implies that the government should consider the difference in discount factors.

This model obtains the optimal monetary policy when the government can commit future policies. However, a natural assumption is that the government cannot commit them. The government, which is elected by naive agents can also be naive. Therefore, the future policy predicted by the current government differs from the policy implemented at that time. This is an important issue for future research.

7 Appendices

A Proof of Proposition 1

First, we solve the problem (17) to obtain the prediction of the future behavior of the agents.

We guess the value functions, (18) and (19), as follows:

\begin{align}
V_c(k, m, \bar{k}, \bar{m}) &= B_c + \ln(m + \mu \bar{m}) - \ln \bar{m} + D_c \ln \bar{k} + E_c \ln(k + F\bar{k}), \quad (A.1) \\
V_l(k, m, \bar{k}, \bar{m}) &= B_l + D_l \ln \bar{k} + E_l \ln(k + F\bar{k}), \quad (A.2)
\end{align}

where \( B_j \), \( D_j \), \( E_j \) (\( j \in c, l \)) and \( F \) are coefficients to be determined. The predicted state
values after the second period from the initial period are \( k_p, m_p, \bar{k}_p, \) and \( \bar{m}_p \). Therefore, the predicted value function after the second period is expressed as \( V_j(k_p, m_p, \bar{k}_p, \bar{m}_p) \).

Hereafter, we discuss the case of the predicted state variables after the second period. Of course, the following discussion also holds for the optimization after the first period from the initial period if we replace \((k_p, m_p, \bar{k}_p, \bar{m}_p)\) with the current state variables \((k, m, \bar{k}, \bar{m})\) (see (A.3) and (A.28)).

Using (A.1) and (A.2), we can rewrite (17) as follows:

\[
V(k_p, m_p, \bar{k}_p, \bar{m}_p) = \max_{c_p, l_p, k_p', m_p'} \left[ \beta_c \ln c_p + \beta_l \theta (1 - l_p) + \beta_c^2 \{ B_c + \ln (m_p' + \mu \bar{m}_p') - \ln \bar{m}_p' + D_c \ln \bar{k}_p' + E_c \ln (k_p' + F \bar{k}_p') \} + \beta_l^2 \{ B_l + D_l \ln \bar{k}_p' + E_l \ln (k_p' + F \bar{k}_p') \} \right]. \tag{A.3}
\]

Based on Assumption 2, that is the agent predicts that the future CIA constraint is binding, the predicted budget constraint is given by \( m_p' = r_p k_p + w_p l_p - k_p' \). When this holds, we can rewrite (21), (22) and (23) as follows:

\[
- \frac{\beta_l \theta}{1 - l_p} + \frac{\beta_c^2 w_p}{r_p k_p + w_p l_p - k_p' + \mu \bar{m}_p'} = 0, \tag{A.4}
\]
\[
- \frac{\beta_c^2}{r_p k_p + w_p l_p - k_p' + \mu \bar{m}_p'} + \frac{\mathcal{E}}{k_p' + F \bar{k}_p'} = 0, \tag{A.5}
\]

where \( \mathcal{E} \equiv \sum_j \beta_j^2 E_j, \ j \in \{c, l\}. \) \tag{A.6}

From Definition 1, the following holds: \( k_p = \bar{k}_p, l_p = \bar{l}_p \) and \( m_p = \bar{m}_p \). Because the future CIA constraints bind, the predicted aggregate consumption is given by \( \bar{m}_p' = \bar{c}_p \).

Substituting these into (A.5) and using (2), (3), and (4), we obtain

\[
\bar{k}_p' = \frac{\mathcal{E}(1 + \mu)}{\beta_c^2 (1 + F) + \mathcal{E}(1 + \mu)} A \hat{\kappa}_p^\alpha \hat{l}_p^{1-\alpha}. \tag{A.7}
\]
Since \( \tilde{m}_p' = \tilde{c}_p = A\bar{k}_p^{\alpha}t_p^{1-\alpha} - \bar{k}_p' \), we obtain

\[
\tilde{m}_p' = \frac{\beta_c^2 (1 + F)}{\beta_c^2 (1 + F) + \mathcal{E}(1 + \mu)} A\bar{k}_p^{\alpha}t_p^{1-\alpha}. \tag{A.8}
\]

Solving (A.5) for \( \bar{k}_p' \), we obtain

\[
k_p' = \frac{\mathcal{E}(r_p k_p + w_p l_p + \mu \tilde{m}_p') - \beta_c^2 F \bar{k}_p'}{\beta_c^2 + \mathcal{E}}. \tag{A.9}
\]

Then, from (A.9), we obtain

\[
k_p' + F \bar{k}_p' = \frac{\mathcal{E}}{\beta_c^2 + \mathcal{E}} W_p, \tag{A.10}
\]

where \( W_p = r_p k_p + w_p l_p + F \bar{k}_p' + \mu \tilde{m}_p' \).

Substituting (A.9) into the budget constraint, \( m_p' = r_p k_p + w_p l_p - k_p' \), we obtain

\[
m_p' = \frac{\beta_c^2 (r_p k_p + w_p l_p + F \bar{k}_p') - \mathcal{E}\mu \tilde{m}_p'}{\beta_c^2 + \mathcal{E}}. \tag{A.11}
\]

Then, using (A.11), we obtain

\[
m_p' + \mu \tilde{m}_p' = \frac{\beta_c^2}{\beta_c^2 + \mathcal{E}} W_p. \tag{A.12}
\]

Using (A.4), (A.5) and (A.10), we obtain

\[
w_p l_p = \frac{(\beta_c^2 + \mathcal{E})w_p - \beta \theta (r_p k_p + F \bar{k}_p' + \mu \tilde{m}_p')}{\beta \theta + \beta_c^2 + \mathcal{E}}. \tag{A.13}
\]

Substituting (A.13) into \( W_p \), we obtain

\[
W_p = \frac{\beta_c^2 + \mathcal{E}}{\beta \theta + \beta_c^2 + \mathcal{E}} (r_p k_p + w_p + F \bar{k}_p' + \mu \tilde{m}_p') \tag{A.14}
\]
Using (A.13) and (A.14), we obtain

\[ w_p(1 - l_p) = \frac{\beta_l \theta}{\beta_c^2 + \mathcal{E}} W_p. \]  \hspace{1cm} (A.15)

Substituting \( k_p = \bar{k}_p \) and \( l_p = \bar{l}_p \) into (A.10) and combining it with (A.4) and (A.5), we obtain

\[ \frac{\beta_l \theta}{1 - l_p} = \frac{\mathcal{E} w_p}{(1 + F) k'_p} \]

Substituting (4) and (A.7) into this equation and solving it for \( \bar{l}_p \), we obtain

\[ \bar{l}_p = \frac{\beta^2(1 - \alpha)}{\beta^2(1 - \alpha) + \beta \theta (1 + \mu)(1 - s_p)}, \]  \hspace{1cm} (A.16)

where \( s_p \equiv \frac{E(1 + \mu)}{\beta^2(1 + F) + \mathcal{E}(1 + \mu)}, \)  \hspace{1cm} (A.17)

Next we determine the coefficients, \( E_j \ (j = c, l) \) and \( F \). Because (A.1) must satisfy (18), the following must hold:

\[
B_c + \ln(m_p + \mu \bar{m}_p) - \ln \bar{m}_p + D_c \ln \bar{k}_p + E_c \ln(k_p + F \bar{k}_p)
\]
\[
= \ln \frac{m + \mu \bar{m}}{1 + \pi_p} + \beta_c [B_c + \ln(m'_p + \mu \bar{m'}_p) - \ln \bar{m'}_p + D_c \ln \bar{k'_p} + E_c \ln(k'_p + F \bar{k'}_p)]
\]
\[
= \beta_c(1 + E_c) \ln W_p + \text{other terms}, \]  \hspace{1cm} (A.18)

where the last equality holds due to (A.10) and (A.12). \textit{Other terms} consist of the terms that have \( \ln(m_p + \mu \bar{m}_p) \), \( \ln \bar{k}_p \), \( \ln \bar{m}_p \) and the parameters. From (3), (4), (A.7), (A.8), and (A.17), (A.14) can be rewritten

\[ W_p = \frac{\beta^2_c + \mathcal{E}}{\beta_l \theta + \beta^2_c + \mathcal{E}} \left[ k_p + \left( \frac{1 - \alpha}{\alpha l_p} + \frac{F s_p}{\alpha} + \frac{\mu(1 - s_p)}{\alpha} \right) \bar{k}_p \right]. \]  \hspace{1cm} (A.19)
Substituting this into (A.18), we obtain

\[ E_c \ln(k_p + F\bar{k}_p) + \text{other terms} \]
\[ = \beta_c(1 + E_c) \ln \left[ k_p + \left( \frac{1 - \alpha}{\alpha l_p} + \frac{F s_p}{\alpha} + \frac{\mu(1 - s_p)}{\alpha} \right) \bar{k}_p \right] + \text{other terms.} \quad (A.20) \]

As the coefficients \( E_c \) and \( \beta_c(1 + E_c) \) must be equal, we obtain

\[ E_c = \frac{\beta_c}{1 - \beta_c}. \quad (A.21) \]

In contrast, because (A.2) must satisfy (19), the following must hold in a similar way to (A.20):

\[ E_l \ln(k_p + F\bar{k}_p) + \text{other terms} \]
\[ = (\theta + \beta_l E_l) \ln \left[ k_p + \left( \frac{1 - \alpha}{\alpha l_p} + \frac{F s_p}{\alpha} + \frac{\mu(1 - s_p)}{\alpha} \right) \bar{k}_p \right] + \text{other terms.} \quad (A.22) \]

As the coefficients \( E_l \) and \( \theta + \beta_l E_l \) must be equal, we obtain

\[ E_l = \frac{\theta}{1 - \beta_l}. \quad (A.23) \]

Substituting (A.21) and (A.23) into (A.6), we obtain

\[ \mathcal{E} = \frac{\beta_c^3}{1 - \beta_c} + \theta \frac{\beta_l^2}{1 - \beta_l}. \quad (A.24) \]

Finally, we determine \( F \). From (A.20) or (A.22), the following must hold

\[ F = \frac{1 - \alpha}{\alpha l_p} + \frac{F s_p}{\alpha} + \frac{\mu(1 - s_p)}{\alpha} \]

Substituting (A.16) and (A.17) into this, we obtain

\[ F = \frac{\beta_c^2 + \beta_l \theta + \mathcal{E}}{\alpha[\beta_c^2(1 + F) + \mathcal{E}(1 + \mu)]}(1 + \mu)(1 + F) - 1. \quad (A.25) \]
Solving this equation for $F$, we obtain

$$F = \frac{\beta_c^2 + \beta \theta + (1 - \alpha) \mathcal{E}}{\alpha \beta_c^2} (1 + \mu) - 1$$

$$= \frac{\beta_c^2 + \theta \beta_c (1 - 1) - \alpha \left( \frac{\beta_c^2}{1 - \beta_c} + \theta \frac{\beta_c}{1 - \beta_c} \right)}{\alpha \beta_c^2} (1 + \mu) - 1 \quad \text{(A.26)}$$

We thus have obtained the coefficients, $E_c$, $E_l$, and $F$ which are to be undetermined. Substituting them into (A.17), we obtain

$$s_p = \frac{\alpha \mathcal{E}}{\beta_c^2 + \beta \theta + \mathcal{E}}. \quad \text{(A.27)}$$

Substituting $E_c$, $E_l$, $F$ and $s_p$ into the equations which determine the endogenous variables $(k_p', m_p', l_p, \bar{c}_p, \bar{k}_p', \bar{m}_p'$ and $\bar{c}_p)$, we obtain the future behavior.

Next, we solve the agent’s problem at the initial time in the BE. Substituting (A.1) and (A.2) into (17), we can rewrite (12) as follows

$$V_0(k, m, \bar{k}, \bar{m}) = \max_{c, l, k', m'} \left[ \ln c + \theta \ln(1 - l) ight. \\
+ \frac{\beta_c}{c} \left\{ B + \ln(m' + \mu m') - \ln \bar{m}' + D_c \ln \bar{k}' + E_c \ln(k' + F \bar{k}') \right\} \\
+ \frac{\beta_l}{l} \left\{ B_l + D_l \ln \bar{k}' + E_l \ln(k' + F \bar{k}') \right\}.$$

Note that $k' = k_p'$, $m' = m_p'$, $\bar{k}' = \bar{k}_p'$ and $\bar{m}' = \bar{m}_p'$ in the initial time because the initial self can determine how much capital and money he/she accumulates. Then, the first order conditions of maximization are given by

$$c : \frac{1}{c} - \lambda - \lambda_{CIA} = 0, \quad \text{(A.29)}$$

$$l : -\frac{\theta}{1 - l} + w \lambda = 0, \quad \text{(A.30)}$$

$$k' : -\lambda + \frac{\beta_c}{k' + F \bar{k}'} = 0, \quad \text{(A.31)}$$

$$m' : -\lambda + \frac{\beta_c}{m' + \mu \bar{m}'} = 0. \quad \text{(A.32)}$$
As the CIA constraint is binding in this equilibrium, \( \bar{c} = \bar{m}' \). Combining (A.31), (A.32), \( \bar{c} = \bar{m}' \) and the market clearing condition (2), we obtain

\[
\bar{k}' = \frac{1}{\beta_c} \left( \frac{\beta_2}{1 - \beta_c} + \theta \frac{\beta_1}{1 - \beta_1} \right) \left( 1 + \mu \right) A \bar{k}^\alpha \bar{l}^{1 - \alpha} \\
= \alpha \beta_c \frac{\beta_2^2}{1 - \beta_c} + \theta \frac{\beta_1}{1 - \beta_1} - \frac{\alpha \theta \beta_1}{1 - \beta_1} (\beta_l - \beta_c) \bar{k}^\alpha \bar{l}^{1 - \alpha}
\]  
(A.33)

The definition of a saving rate is \( \text{investment}/\text{output} \). Therefore,

\[
s^* = \alpha \beta_c \frac{\beta_2}{1 - \beta_c} + \theta \frac{\beta_1}{1 - \beta_1} - \frac{\alpha \theta \beta_1}{1 - \beta_1} (\beta_l - \beta_c).
\]  
(A.34)

Combining (A.30), (A.32) and \( \bar{c} = \bar{m}' = (1 - s^*)A \bar{k}^\alpha (\bar{l}^\star)^{1 - \alpha} \), we have

\[
\bar{l}^\star = \frac{1 - \alpha}{1 - \alpha + \theta (1 - s^*)^{1 + \mu} \bar{\beta}_c^\alpha}.
\]  
(A.35)

Moreover, from (5), we obtain

\[
\pi^* = \frac{(1 + \mu) \bar{m}}{\bar{m}'} - 1 = \frac{(1 + \mu) \bar{m}}{(1 - s^*) A \bar{k}^\alpha (\bar{l}^\star)^{1 - \alpha}} - 1.
\]

At the end of this appendix, we show that there exists a lower bound of \( \mu \); that is \( \mu \).

We have assumed that the future predicted CIA constraint is binding (See Assumption 2). If so, the predicted real gross interest rate must not be lower than the predicted real return of money holdings, that is,

\[
r_p \geq \frac{1}{1 + \pi_p}
\]  
(A.36)

because there is no incentive to hold capital when the predicted gross interest rate is lower than the return of money holdings. If the predicted CIA constraint is binding, \( \bar{m}' = \bar{c}_p \).
From (A.9) and (A.17), we obtain

\[ \bar{k}_p' = s_p A \bar{k}^{\alpha_l \bar{l}_p}_{\bar{l}_p}. \tag{A.37} \]

Substituting this into (2), we obtain

\[ \bar{m}_p' = \bar{c}_p = (1 - s_p) A \bar{k}^{\alpha_l \bar{l}_p}_{\bar{l}_p}. \tag{A.38} \]

Combing (5) and (A.38), we obtain the following predicted inflation rate\(^{12}\):

\[ 1 + \pi_p = \frac{(1 + \mu) \bar{m}_p}{(1 - s_p) A \bar{k}^{\alpha_l \bar{l}_p}_{\bar{l}_p}}. \tag{A.39} \]

Using (3) and (A.39), we can rewrite (A.36) as follows:

\[ r_p = \alpha A \bar{k}^{\alpha_l \bar{l}_p}_{\bar{l}_p} \geq \frac{(1 - s_p) A \bar{k}^{\alpha_l \bar{l}_p}_{\bar{l}_p}}{(1 + \mu) \bar{m}_p} = \frac{1}{1 + \pi_p}, \]

\[ \Rightarrow \quad \alpha \frac{1 + \mu}{1 - s_p} \bar{m}_p \geq \bar{k}_p \tag{A.40} \]

This relationship must also be satisfied in the next period. In other words, we can replace \( \bar{m}_p \) and \( \bar{k}_p \) with \( \bar{m}_p' \) and \( \bar{k}_p' \). Therefore, from (A.37) and (A.38), we can rewrite (A.40) as follows:

\[ \alpha \frac{1 + \mu}{1 - s_p} (1 - s_p) A \bar{k}^{\alpha_l \bar{l}_p}_{\bar{l}_p} \geq s_p A \bar{k}^{\alpha_l \bar{l}_p}_{\bar{l}_p}, \]

\[ \Rightarrow \quad 1 + \mu \geq \frac{s_p}{\alpha} = \frac{\beta_c^2}{1 - \beta_c} + \theta \frac{\beta_c^2}{1 - \beta_c} \tag{A.41} \]

where we substitute (A.27) into \( s_p \). Note that (A.41) satisfies the definition of the agent’s prediction (Definition 1). Therefore, this inequality defines the lower bound of \( 1 + \mu \), that is, \( 1 + \mu_\). Therefore, we have shown Proposition 1.

\(^{12}\)(A.39) is an increasing function of \( \mu \) because \( \frac{\partial s_p}{\partial \mu} = 0 \) and \( \frac{\partial \bar{k}_p}{\partial \mu} < 0 \) from (A.16) and (A.27).
B Proof of Proposition 2

We can use the first order conditions (A.29)-(A.32); however, the Lagrange multiplier associated with the CIA constraint, $\lambda_{CIA}$, is equal to 0 in the NBE. As the future value function is the same as (A.3) in Appendix A, we can apply (A.21) and (A.23) for this problem. Note that, at the symmetric equilibrium, $l = \bar{l}$, $k = \bar{k}$ and $m = \bar{m}$. Using (2), (A.29) and (A.31), we obtain

$$k^{**} = \frac{\beta_c^2}{1 - \beta_c} + \theta \frac{\beta_i}{1 - \beta_i} A\bar{k}^{\alpha} \bar{l}^{1-\alpha}, \quad \text{(B.1)}$$

$$\bar{c}^{**} = \frac{1 + F + \beta_c^2}{1 + F + \beta_c^2 + \theta \beta_i} A\bar{k}^{\alpha} \bar{l}^{1-\alpha}, \quad \text{(B.2)}$$

From (B.1), the saving rate of the NBE is given by

$$s^{**} = \frac{\beta_c^2 + \theta \beta_i}{1 + F + \beta_c^2 + \theta \beta_i}.$$ \quad \text{(B.3)}$$

Combining (A.29), (A.30), (B.2) and the market clearing conditions, we can obtain the labor supply:

$$\bar{l}^{**} = \frac{1 - \alpha}{1 - \alpha + \theta (1 - s^{**})}. \quad \text{(B.4)}$$

Moreover, combining (A.31), (A.32), (B.1), and (2), we obtain the real money balance in the next period:

$$m^{**} = \frac{\beta_c (1 + F)}{(1 + F + \beta_c^2 + \theta \beta_i)(1 + \mu)} A\bar{k}^{\alpha} \bar{l}^{1-\alpha}. \quad \text{(B.5)}$$

Therefore, we have obtained the saving rate, labor supply, and real money balance in the next period in Proposition 2.

Next, we show that the NBE obtains when $1 + \mu < \beta_c$. If the CIA constraint (11) is not binding, then $\bar{m} > \bar{c}$. From (2), (B.2), and (B.5), when $\bar{m} > \bar{c}$, $1 + \mu < \beta_c$ holds.

The reason why $1 + \mu < 1 + \mu$ is to bind the predicted CIA constraint. It is the same
as Proposition 1.

References


