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The State Finance and Public Goods from The General Equilibrium Viewpoint: Fundamental Welfare Theorems for Lindahlian General Equilibrium with Money*

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Abstract

This paper deals with the concept of *Lindahl general equilibrium with money* and its relation to the first and second fundamental theorems of welfare economics and other problems like core arguments. This paper is concerned with the question of how the optimal supply of public goods can be achieved in an *ideal situation* based on market transactions. The paper also points out that unless the (Lindahlian) financing problem is treated together with the fiscal problem of the state (as a monetary equilibrium), the fundamental theorems of economics may lead to the wrong message in terms of policy. The *ideal state* described in this paper is one in which public and private firms and their profits are clearly distinguished, and in which the optimal allocation is *feasible only under a constant government deficit.* This is not addressed in the traditional Lindahl general equilibrium and cost-sharing equilibrium. Also, under such a general situation, the activity criterion for public firms in the *ideal state* is organized as a special type of *marginal cost pricing.* The framework of this paper, including government activities (optimal money issuance and taxation), is of extremely urgent significance, especially as seen in the recent problems of health care costs and state finances.

Keywords: Lindahl Equilibrium, General Equilibrium, Fundamental Theorems of Welfare Economics, Public Goods, Satiation, Money

JEL Classification: D51, D62, H53

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1 Introduction

In this paper, we consider the problem of the state finance (including the issuance of fiat money) and public goods from the general equilibrium perspective (the Lindahl general equilibrium with income transfers). Although it is usually difficult to present a realistic implication of Lindahl equilibria, with respect to merit goods like medical cares, it would be available to bring about arguments on an idealistic fiscal state of government. Here, the government is treated as the supplier of fiat money as well as a provider of public goods. In such a setting, several important, previously unknown results can be obtained regarding the government's fiscal balance, the roles and norms of public and private firms, and especially the most desirable way to share the costs of public goods.

The general equilibrium argument including public goods begins with Foley (1970) and extended by Milleron (1972) to the case with multiple firms. These arguments, however, do not allow firms to have non-convex technologies. Mas-Colell and Silvestre (1989) treats the case with the non-convex technology though their arguments depend on a special cost-function structure and do not incorporate multiple firm settings. In our model, we treat public and private multiple firms as well as deficits of the government because of the non-convexity of the public-firm technology in order to describe an ideal resource allocation (to establish the first and second welfare results) for public-goods economies.

One of the most important purposes of our model is to describe a situation such that a certain activity arrangement that is necessary for Pareto optimality can never be realized as an equilibrium until the government prepares for a budgetary deficit. The problem is related, for one thing, to the *saturation* of consumers' preferences (or to the necessity of savings and capital accumulations in view of *dynamics*). The equilibrium concept in this paper is generally called a *dividend equilibrium* (competitive equilibrium with *slack* in Mas-Colell 1992), in which government fiat money often plays an essential role. At the same time, since the public-firm technology is not necessarily convex, we have to present a criterion for the public-firm production decision alternative to the profit maximization. Let us first review these points with the following simple example:

> **(Example 1)** Suppose that there are two commodities, a public good (commodity 1) and a private good (commodity 2). Here the public good is assumed to have both the non-excludability and non-rivalness. Consider a public firm and a private firm, and two consumers (a worker and a capitalist). The worker (i_w) has one unit of private good (identified with one unit of labour) as the initial endowment. The capitalist (i_c) owns the private firm (technology $Y_1 \subset R^2$) and having no private good. Government possesses the public technology $Y_0 \subset R^2$. Each consumer has consumption set R^2_+ with a utility function such that $u(x_1, x_2) = \max\{x_2, 8\}$. (That is, each consumer has a satiated preference and does not obtain positive utility from the public good.) Furthermore, suppose that $Y_0 = \{(0,0), (-1,1)\}$ and $Y_1 = \{(y_1, y_2) \mid y_2 = \frac{\sqrt{10}}{10}\sqrt{10 - y_1} - 1,\ 0 \leqq y_1 \leqq 10\}$. In other words, Y_0 is a technology that transforms one unit of private good into one unit of public good (or does nothing), and Y_1 is a well behaved technology that transforms one unit of public good into 10 units of private good (see Figure 1).

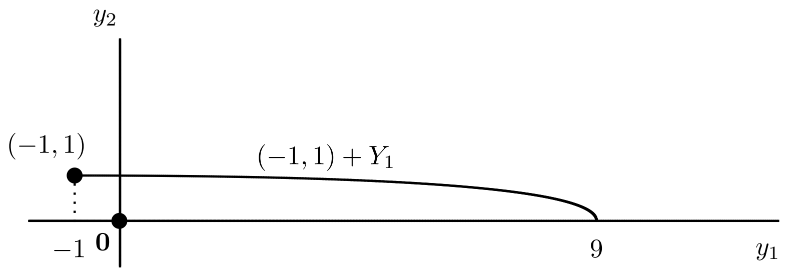

Figure 1: $Y_0 = \{(0,0),(-1,1)\}$ and $(-1,1)+Y_1$

In the setting of Example 1 above, it is clear that the production of the public good must take place for Pareto optimality. Furthermore, for the competitive private firm, only $p = (p_1, p_2) = (0,1)$ is the price (with norm 1) to support production behavior that produces 10 units of private good with one unit of public good input, while satisfying the profit-maximization condition. It follows that if we do not use any income-transfer policy, the unique competitive (Lindahl) equilibrium, where dividends $d_w = 0$ for worker and $d_c = 10$ for capitalist (the profit of private firm), is the allocation, $x^w = (1,1)$ for worker and $x^c = (1,9)$ for capitalist.[1] The allocation is not Pareto optimal involving a government budget deficit $-1 = p \cdot (-1,1)$, even though the result is better than a situation where no production occurs. If we allow the government to further increase its budget deficit and allow a positive dividend (fiat money issuance) $d^w = 1$ to worker, we obtain a Pareto optimal (dividend) competitive equilibrium allocation, $x^w = (1,2)$ and $x^c = (1,8)$, with a government budget deficit -2. For these Lindahl general equilibrium with dividends, the satiation of preference of capitalist or, from a dynamic perspective, savings (especially under the overlapping-generations framework) is essential.[2] The savings of capitalist is equal to the government budget deficit for both cases. Such a government deficit can be viewed as the loan from the private sector, i.e., credit given to the state from the private sector. In fact, the private sector's faith in the value of its savings is precisely the same as its faith in the state, money, and the market system. Can we really say that such a government deficit is a bad thing and must be eliminated? Moreover, it is only with such a government deficit (unless we assume a lump-sum tax from the private firm) that a Pareto-optimal allocation of resources can be realized.

This paper deals with the first and second fundamental theorems of welfare economics for (Lindahl) market equilibrium with public goods, including the distinction between public and private firms, as well as the government's ability to issue money (credit). Such a treatment is important in today's society, where public goods (as merit goods) such like medical cares, account for a large share of state finances through public and private institutions. In particular, our model provides a description of an ideal state of cost-sharing for public goods, while taking into account what the state's contribution is and what the alternative norm is for profit maximization in a public firm.[3]

[1] Note that the price of public good is 0 and the preferences are satiated, so $u(1,9) = u(1,8)$ maximizes the utility of capitalist.

[2] See Appendix A.

[3] As emphasized in the above example, the essential role played by the satiation in a static model can be replaced by

Denote by R the set of real numbers. For finite set A, $\sharp A$ denotes the number of elements of A. We write R^A instead of $R^{\sharp A}$ to represent the $\sharp A$-dimensional topological vector space. Order relations on R^A, $\geqq$ and $>$, are defined respectively as $(x_a)_{a\in A} \geqq (y_a)_{a\in A}$ iff $x_a \geqq y_a$ for all $a \in A$, and $(x_a)_{a\in A} > (y_a)_{a\in A}$ iff $(x_a)_{a\in A} \geqq (y_a)_{a\in A}$ and $(x_a)_{a\in A} \neq (y_a)_{a\in A}$. We also define relation $\gg$ as $(x_a)_{a\in A} \gg (y_a)_{a\in A}$ iff $x_a > y_a$ for all $a \in A$. By R^A_+ and R^A_{++}, we represent the sets $\{x \in R^A \mid x \geqq 0\}$ and $\{x \in R^A \mid x \gg 0\}$, respectively.

2 The Model

2.1 Basic Settings

Denote by $K \cup L$, the finite set of commodities, where $K \neq \emptyset$ is the index set of *public goods* and $L \neq \emptyset$ is the set of *private goods* ($K \cap L = \emptyset$). As in Foley (1970), a vector of public and private goods is written as $((x_k)_{k\in K}; (z_\ell)_{\ell\in L}) = (x; z) \in R^{K\cup L}$. Notation I is used to denote the non-empty index set of finite agents. Each agent $i \in I$ is represented by $(\succsim_i, \omega_i)$, where $\succsim_i$ is the preference relation on consumption set $X_i = R^{K\cup L}_+$ and $\omega_i \in R^L$ is the initial endowment. The preferences are assumed to satisfy reflexivity, transitivity, completeness and continuity. The preferences, therefore, can be represented by utility functions. Note, however, that the preferences are allowed to be satiated.

An economy, $\mathcal{E}$, is identified with a finite list of consumers, $(\succsim_i, \omega_i)_{i\in I}$, a finite list of private firm (convex) technologies, $(Y_j)_{j\in J}$ and a public firm (possibly non-convex) technology Y_0. As Milleron (1972), technologies are identified with subsets of $R^K_+ \times R^K_+ \times R^L$, where $(x^-, x^+; y) \in R^K_+ \times R^K_+ \times R^L$ means that x^- is an input vector of public goods, x^+ is an output vector of public goods, and y is a net production vector of private goods. (We consider the situation that output and input prices are different for public goods.) We assume the following:

(1-1) For each $j \in J$, technology Y_j is a closed subset of $R^K_+ \times R^K_+ \times R^L$ having 0 as its element.

(1-2) Technology Y_0 is a non-empty closed (possibly non-convex) subset of $R^K_+ \times R^K_+ \times R^L$.

Note that we use the most general framework including multi-firms like Milleron (1972) and positive profits like Mas-Colell and Silvestre (1989), while not using a special condition for public good inputs like Foley (1970)'s condition (B.5) "No public good is necessary as a production input".

By the meaning of public goods, we may view the set of all technically possible production plans in the economy, Y, as the following set.

$$Y = \{(x; y) \mid x = \sum_{j\in J\cup\{0\}} x^+_j, y = \sum_{j\in J\cup\{0\}} y_j, (x, x^+_j; y_j) \in Y_j \text{ for all } j \in J \cup \{0\}\}. \quad (1)$$

such things as savings under a dynamic overlapping-generations structure (see, Appendix A). In other words, the most important question is how much savings (the value left by someone) contribute to society, taking into account production (including the future), and the appropriate government budget deficit should be determined on this basis.

For firms, we also assume that there is a list of *shareholding rates*, $((\theta_{ij})_{j\in J})_{i\in I}$.

Technology Y is used to define a feasible allocation of economy $\mathcal{E}$. A *consumption allocation* for a list of agents $S \subset I$ is a sequence of elements of consumption sets, $((x_i; z_i) \in R_+^{K\cup L})_{i\in S}$. For an economy $\mathcal{E} = ((\succsim_i, \omega_i)_{i\in I}, Y_0, (Y_j)_{j\in J}, ((\theta_{ij})_{j\in J})_{i\in I})$, if consumption allocation $((x; z_i) \in R_+^{K\cup L})_{i\in I}$ satisfies

$$\sum_{i\in I} z_i = \sum_{i\in I} \omega_i + y \text{ for some } (x; y) \in Y, \tag{2}$$

we say that $(x; z_i)_{i\in I}$ is *feasible* under $(x; y) \in Y$. The list of production actions associated with $(x; y)$, i.e., $\big((x, x_0^+; y_0) \in Y_0, ((x, x_j^+; y_j) \in Y_j)_{j\in J}\big)$, where $y = y_0 + \sum_{j\in J} y_j$, is called a *production allocation*. We call the $(m+n+1)$-tuple $((x; z_i)_{i\in I}, (x, x_0^+; y_0), (x, x_j^+; y_j)_{j\in J})$ a *state* of $\mathcal{E}$.

2.2 Lindahl Equilibrium with Income Transfers

The list of price vectors, $(p_K^*, p_L^*) \in R^{K\cup L}$, input price vectors for consumers, $(p_I^{i\,*} \in R^K)_{i\in I}$, producers, $(p_J^{j\,*} \in R^K)_{j\in J}$, and public firm, $p^{0*} \in R^K$, such that $p_K^* = p^{0*} + \sum_{i\in I} p_I^{i\,*} + \sum_{j\in J} p_J^{j\,*}$, income transfers to consumers, $(d_i^*)_{i\in I} \in R^I$, and feasible consumption allocation, $(x^*; z_i^*)_{i\in I}$ under $(x^*; y^*) \in Y$ with $y^* = \sum_{j\in J\cup\{0\}} y_j^*$, where $(x^*, x_0^{*+}; y_0^*) \in Y_0$ and $(x^*, x_j^{*+}; y_j^*) \in Y_j$ for each $j \in J$, is called a *Lindahl quasi-equilibrium* with income transfers, $(p_K^*, p_L^*, (p_I^{i\,*})_{i\in I}, (p_J^{j\,*})_{j\in J}, p^{0*}, (d_i^*)_{i\in I})$, for $\mathcal{E} = ((\succsim_i, \omega_i)_{i\in I}, Y_0, (Y_j)_{j\in J}, ((\theta_{ij})_{j\in J})_{i\in I})$, if the following conditions are satisfied.

(i) Profit maximization for private firms: $p_K^* \cdot x_j^{*+} - p_J^{j\,*} \cdot x^* + p_L^* \cdot y_j^* \geqq p_K^* \cdot x_j^+ - p_J^{j\,*} \cdot x_j^- + p_L^* \cdot y_j$ for all $(x_j^-, x_j^+; y_j) \in Y_j$ for each $j \in J$. We denote by π_j^* the profit of firm j (the value of the left hand side of this inequality) for each $j \in J$.

(ii) Profit Maximality for the public firm: $p_K^* \cdot x_0^{*+} - p^{0*} \cdot x^* + p_L^* \cdot y_0^* \geqq p_K^* \cdot x_0^+ - p^{0*} \cdot x_0^- + p_L^* \cdot y_0$ for all $(x_0^-, x_0^+; y_0) \in Y_0$ such that $(x_0^-, x_0^+; y_0)$ is associated with an allocation that is Pareto superior to the status quo allocation.[4] We denote by π_0^* the profit of the public firm (the value of the left hand side of this inequality).

(iii) Expenditure minimization for consumers: for each $i \in I$, $(x^*; z_i^*)$ satisfies $p_I^{i\,*} \cdot x^* + p_L^* \cdot z_i^* \leqq p_L^* \cdot \omega_i + \sum_{j\in J} \theta_{ij}\pi_j^* + d_i^*$ and for all $(x; z_i)$ such that $(x; z_i) \succsim_i (x^*; z_i^*)$, we have $p_I^{i\,*} \cdot x + p_L^* \cdot z_i \geqq p_I^{i\,*} \cdot x^* + p_L^* \cdot z_i^*$.

The expenditure minimization implies the following utility maximization if we can redefine d_i^* so as to satisfy $p_I^{i\,*} \cdot x^* + p_L^* \cdot z_i^* = p_L^* \cdot \omega_i + \sum_{j\in J} \theta_{ij}\pi_j^* + d_i^*$ and the *minimum wealth condition*, $\inf\{p_I^{i\,*} \cdot x + p_L^* \cdot z_i \,|\, (x; z) \in X_i\} < p_L^* \cdot \omega_i + \sum_{j\in J} \theta_{ij}\pi_j^* + d_i^*$, is satisfied (Debreu 1959[p.69,(1)]).

(iv) Utility maximization for consumers: for each $i \in I$, $(x^*; z_i^*)$ is the $\succsim_i$-greatest element of $\{(x; z_i) \in R^{K\cup L} \mid p_I^{i\,*} \cdot x + p_L^* \cdot z_i \leqq p_L^* \cdot \omega_i + \sum_{j\in J} \theta_{ij}\pi_j^* + d_i^*\}$.

[4] The condition is automatically satisfied under the ordinary profit maximization condition.

If conditions (i), (ii), and (iv) instead of (iii) are satisfied, the same list of prices and feasible allocations for $\boldsymbol{\mathcal{E}}$ is called a *Lindahl equilibrium* with income transfers. It follows that if there exists a Lindahl quasi-equilibrium, the allocation and the Lindahl price system can also be identified with a Lindahl equilibrium by redefining each $d_i \in R$ to satisfy the equation,

$$p_I^{i*} \cdot x^* + p_L^* \cdot z_i^* = p_L^* \cdot \omega_i + \sum_{j \in J} \theta_{ij} \pi_j^* + d_i^*, \tag{3}$$

as long as the minimum wealth condition, $\inf\{p_I^{i*} \cdot x + p_L^* \cdot z_i | (x; z) \in X_i\} < p_L^* \cdot \omega_i + \sum_{j \in J} \theta_{ij} \pi_j^* + d_i^*$, for every $i \in I$ is satisfied.

Note that in this paper, we do not use assumptions like local non-satiation that necessarily make equation (3) hold for each i in conditions (1) or (2). This is because several important examples, as seen in the Introduction, are provided along with the existence of satiation points of preferences in an equilibrium state.[5]

If equation (3) holds for all $i \in I$, by incorporating the feasibility equation (2), we have the next lemma.

Lemma 1: *A feasible allocation $(x^*; z_i^*)_{i \in I}$ under $(x^*, y^*) \in Y$ such that equation (3) for all $i \in I$ holds relative to a certain Lindahl price system with income transfers, $(p_K^*, p_L^*, (p_I^{i*})_{i \in I}, (p_J^{j*})_{j \in J}, p^{0*}, (d_i^*)_{i \in I})$, satisfies $\sum_{i \in I} p_I^{i*} \cdot x^* + \sum_{j \in J \cup \{0\}} p_L^* \cdot y_j^* = \sum_{j \in J} \pi_j^* + \sum_{i \in I} d_i^*$. (Without equation (3), $\leqq$ holds.)*

Proof: Summing up equation (3) of each $i \in I$, we have $\sum_{i \in I}(p_I^{i*} \cdot x^* + p_L^* \cdot z_i^*) = \sum_{i \in I}(p_L^* \cdot \omega_i + \sum_{j \in J} \theta_{ij} \pi_j^* + d_i^*)$. By considering the feasibility equation (2) together with $\sum_{i \in I} \theta_{ij} = 1$ for each $j \in J$, we obtain the result. ■

In Lemma 1, the left hand side amount, $\sum_{i \in I} p_I^{i*} \cdot x^* + \sum_{j \in J \cup \{0\}} p_L^* \cdot y_j^*$, can be identified with the sum of all profits of private and public firms.

Lemma 2: *For feasible allocation $(x^*; z_i^*)_{i \in I}$ under $(x^*, y^*) \in Y$, amount $\sum_{j \in J \cup \{0\}} p_L \cdot y_j + \sum_{i \in I} p_I^i \cdot x$ relative to a certain Lindahl price system, $(p_K, p_L, (p_I^i)_{i \in I}, (p_J^j)_{j \in J}, p^0)$, is equal to the sum of all profits of private and public firms, $\sum_{j \in J}(p_K \cdot x_j^+ - p_J^j \cdot x + p_L \cdot y_j) + (p_K \cdot x_0^+ - p^0 \cdot x + p_L \cdot y_0)$.*

Proof: We may calculate the sum of profits of private and public firms as follows.

$$\begin{aligned}
& \sum_{j \in J}(p_K \cdot x_j^+ - p_J^j \cdot x + p_L \cdot y_j) + (p_K \cdot x_0^+ - p^0 \cdot x + p_L \cdot y_0) \\
= & \sum_{j \in J}(p_K \cdot x_j^+ - p_J^j \cdot x + p_L \cdot y_j) + (\sum_{i \in I} p_I^i + \sum_{j \in J} p_J^j) \cdot x - p_K \cdot \sum_{j \in J} x_j^+ + p_L \cdot y_0 \\
= & \sum_{j \in J}(p_L \cdot y_j - p_J^j \cdot x) + (\sum_{i \in I} p_I^i + \sum_{j \in J} p_J^j) \cdot x + p_L \cdot y_0 \\
= & \sum_{j \in J \cup \{0\}} p_L \cdot y_j + \sum_{i \in I} p_I^i \cdot x
\end{aligned}$$

[5] Usually, assumptions such as local non-satiation are also used to show that condition (iv) means condition (iii). For example, (iv) implies (iii) if preferences are convex and point $(x^*; z_i^*)$ is not a satiation point (Debreu 1959 [p.71,(2)]). We also note that if preferences are strongly convex, every satiation point, if such an exists, is unique, so condition (iii) is automatically satisfied.

In the above, note that the feasibility condition is not necessary. ■

Hence, by Lemma 2, Lemma 1 also means the following.

Lemma 3: *A feasible allocation* $(x^*; z_i^*)_{i\in I}$ *under* $(x^*, y^*) \in Y$ *such that equation (3) for all* $i \in I$ *holds relative to a certain Lindahl price system with income transfers,* $(p_K^*, p_L^*, (p_I^{i*})_{i\in I}, (p_J^{j*})_{j\in J}, p^{0*}, (d_i^*)_{i\in I})$*, satisfies* $\pi_0^* = \sum_{i\in I} d_i^*$. *(Without equation (3),* $\leqq$ *holds.)*

Remark 1 (Wealth Transfer): We may identify $p_L^* \cdot \omega_i + \sum_{j\in J} \theta_{ij}\pi_j^* + d_i$ with the amount that Samuelson (1969) proposed to call a *lump-sum redistribution* because he seems to assume that $\omega_i = 0$ for all i. Foley (1970) also uses the same amount minus $p_L^* \cdot \omega_i$ as a lump-sum transfer, which he assumes to be 0 under the situation like Lemma 1 since he treats only cone-shaped technologies with 0-profits. A positive profit case is treated by Mas-Colell and Silvestre (1989) under a (possibly) non-convex single (aggregate) technology.

Remark 2 (Dividend Equilibrium): Because of the existence of satiation and income transfers together with the rigorous feasibility equation (2), we base our equilibrium concept on a dividend equilibrium that is firstly defined by Aumann and Drèze (1986) for pure exchange economies and extended by Mas-Colell (1992) to production economies. As Kajii (1996) discussed, such dividends or non-negative wealth transfers can also be reinterpreted as *money* (see Murakami and Urai 2017). Urai et al. (2022) further extended the concept to the problem of monetary equilibrium in overlapping generations economies.

Remark 3 (Non-convex Second Welfare Theorem): In this paper, we treat second welfare theorem for economies including public goods, monetary transfers and non-convex technologies. For second welfare problems in economies with non-convex technologies, see, e.g., Kamiya (1995).

Discussion 1 (Lindahl Equilibrium with Transfers and Production Core)

Example 1 in the Introduction also provides an important setting for a production core problem for economies including public goods. In the example, the government's cost of producing the public good is 1. Let such a cost be shared equally (half and half) between the worker and the capitalist. The method is simple: $d_0 = -0.5$ and $d_1 = 10 - 0.5 = 9.5$. Furthermore, suppose that there is no satiation in the consumers' utility. Although the Lindahl equilibrium (worker consumes 0.5 and capitalist consumes 9.5 of the private good) is Pareto optimal, it is not in the core (obviously, blocked by the worker even in the weakest condition that production technologies cannot be used for any deviation). However, it "apparently" satisfies Mas-Colell and Silvestre's sufficient condition for core allocations, that is, "the cost-burden of each agent is non-negative." Here, we need to rethink "each agent's cost-burden" in light of the difference between the settings in Mas-Colell and Silvestre (1989) and ours. In our settings, public and private firms are distinguished, and the public good produced by the public firm only makes sense as an input for the private firm. The profit of the private firm, 10, is calculated, therefore, without deducting the cost necessary for producing the public good. It can be said that the "true" profit that the capitalist should have received is 9, so the fact that he receives a dividend of 9.5 means that "the cost-burden is (essentially) negative for the capitalist." It follows that under an income transfer policy (tax system) that attributes the burden of public-good production to the wrong place, the Lindahl equilibrium is no longer a core allocation.

Discussion 2 (Optimal Tax Policy)

Example 1 and the discussion in Remark 4 shows that in the general situation, where multiple firms exist and public goods are used in production, the "true" cost burden of public-goods production is difficult to determine because of the existence of private profits that free-ride on the action of the public firm. This also means that as long as such free-riding exists through corporate profits, the goal of balanced budgets alone will lead to a non-core Lindahl income transfer equilibrium through a wrong income distribution (wrong tax system and cost-bearing requirements). Conversely, however, we can consider what the optimal tax system (cost-sharing for public goods) is by asking about the income transfer conditions for the Lindahl equilibrium to be a core allocation. Of course, in this case, the optimality of the tax system would depend on the definition of the production core, taking into account a certain "fairness". For example, if all production technologies can be used at the time of deviation, then a fair tax system would be based on the idea that "the knowledge of production technology itself can be used by everyone. If, on the other hand, a portion of the production technology can be used at the time of deviation in proportion to the shareholding ratio, then the fairness of the tax system will take into account the meaning of "intellectual property and shareholding" in the private ownership economy as much as possible. Furthermore, if we add the requirement to "fix the behavior of the public firm" at the time of deviation, then the fair tax system is to be considered with the greatest emphasis on "without free-riding on the current behavior of the public firm." It follows that the problem of optimal taxation is possible to be generalized as a question like the existence of a Lindahl equilibrium with income transfers leading to a core allocation.

3 Fundamental Welfare Theorems for Lindahl Equilibria

Now we provide the first fundamental theorem of welfare economics for Linhahl equilibria. (Condition (iii) must be added here in addition to condition (iv). See footnote 5.)

Theorem 1 (The First Fundamental Theorem): *A Lindahl equilibrium allocation,* $(x^*; z_i^*)_{i\in I}$*, under a system of equilibrium Lindahl price vectors and income transfers,* $(p_K^*, p_L^*, (p_I^{i\,*})_{i\in I}, (p_J^{j\,*})_{j\in J}, p^{0*}, (d_i^*)_{i\in I})$*, is Pareto optimal if condition (iii) is satisfied.*

Proof: Assume the contrary. Then we have a feasible allocation, $(x'; z_i')_{i\in I}$, under $(x'; y') \in Y$ such that $(x'; z_i') \succ_i (x^*; z_i^*)$ at least one $i \in I$ and $(x'; z_i') \succsim_i (x^*; z_i^*)$ for all $i \in I$. By considering the utility maximization condition (iv) and the expenditure minimization condition (iii), we have the following inequality.

$$\sum_{i\in I}(p_I^{i*}\cdot x' + p_L^*\cdot z_i') > \sum_{i\in I}(p_I^{i*}\cdot x^* + p_L^*\cdot z_i^*). \tag{4}$$

Feasibility condition (2) means that the left hand side is equal to $\sum_{i\in I} p_I^{i*}\cdot x' + p_L^*\cdot \sum_{i\in I} z_i' = \sum_{i\in I} p_I^{i*}\cdot x' + p_L^*\cdot(y' + \sum_{i\in I}\omega_i)$. On the other hand, by the same equation, the right hand side is equal to $\sum_{i\in I} p_I^{i*}\cdot x^* + p_L^*\cdot\sum_{i\in I} z_i^* = \sum_{i\in I} p_I^{i*}\cdot x^* + p_L^*\cdot(y^* + \sum_{i\in I}\omega_i)$. It follows that we have,

$$\sum_{i\in I} p_I^{i*}\cdot x' + p_L^*\cdot y' + p_L^*\cdot\sum_{i\in I}\omega_i > \sum_{i\in I} p_I^{i*}\cdot x^* + p_L^*\cdot y^* + p_L^*\cdot\sum_{i\in I}\omega_i, \tag{5}$$

and

$$\sum_{i\in I} p_I^{i*}\cdot x' + p_L^*\cdot y' > \sum_{i\in I} p_I^{i*}\cdot x^* + p_L^*\cdot y^*. \tag{6}$$

By Lemma 2, the both sides of the above inequality are equal to the sum of all profits of private and public firms, which contradicts to the profit maximization and maximality conditions (i) and (ii) of Lindahl equilibria. ■

Next we show the second welfare theorem for Lindahl equilibria. The second theorem is given in the form of quasi-equilibrium as in Debreu (1959; p.95,6.4). As stated before, a quasi-equilibrium is an equilibrium as long as the minimum wealth condition is satisfied.

Theorem 2 (The Second Fundamental Theorem): *Every Pareto optimal allocation, $(x^*; z_i^*)_{i\in I}$, feasible under $(x^*; y^*)\in Y$ with $y^* = y_0^* + \sum_{j\in J} y_j^*$ for $\mathcal{E} = \big((\precsim_i, \omega_i)_{i\in I}, Y_0, (Y_j)_{j\in J}, ((\theta_{ij})_{j\in J})_{i\in I}\big)$, where each Y_j, $j\in J$, is convex and there exists at least one i' such that $(x^*; z_{i'})$ is not satiated, is a Lindahl quasi-equilibirum under price vectors and income transfers, $(p_K^*, p_L^*, (p_I^{i*})_{i\in I}, (p_J^{j*})_{j\in J}, p^{0*}, (d_i^*)_{i\in I})$.*

Proof: As Milleron (1972), we identify in this proof the subspace, R^K, of commodity space $R^{K\cup L}$ with $(\sharp I + \sharp J + 1)$-times replication of R^K in order to describe input vectors of public goods for consumers $i\in I$, private firms $(Y_j)_{j\in J}$, and public firm Y_0. The identification enables us to treat public goods as if they were ordinary private goods at least from the viewpoint of all decision makers. Let $(K_i)_{i\in I}$, $(K_j)_{j\in J}$, and K_0 be disjoint family of $(\sharp I + \sharp J + 1)$-times replication of index set K for public goods, and consider the following extended commodity space $\boldsymbol{E}$ of $R^{K\cup L}$.

$$\boldsymbol{E} = \prod_{i\in I} R^{K_i}\times\prod_{j\in J} R^{K_j}\times R^{K_0}\times R^L. \tag{7}$$

Let us identify Y_0 with subset $\hat{Y}_0$ of $\boldsymbol{E}$ through the next canonical identification mapping:

$$h_0 : Y_0 \ni (x_0^-, x_0^+, y_0) \mapsto (x_0^+, ..., x_0^+, x_0^+ - x_0^-; y_0) \in \hat{Y}_0 \subset \prod_{i\in I} R^{K_i}\times\prod_{j\in J} R^{K_j}\times R^{K_0}\times R^L. \tag{8}$$

In the same way, identify Y_j for each $j\in J$ with $\hat{Y}_j$ by

$$h_j : Y_j \ni (x_j^-, x_j^+, y_j) \mapsto (x_j^+, ..., x_j^+, x_j^+ - x_j^-, x_j^+, ..., x_j^+; y_0) \in \hat{Y}_j \subset \prod_{i\in I} R^{K_i}\times\prod_{j\in J} R^{K_j}\times R^{K_0}\times R^L, \tag{9}$$

where the entry for $x_j^+ - x_j^-$ is the coordinate corresponding to R^{K_j}. Note that for each $j \in J$, the convexity of Y_j clearly implies the convexity of $\hat{Y}_j$. Moreover, identify X_i for each $i \in I$ with its image through the next mapping,

$$h_i : X_i \ni (x_i; z_i) \mapsto (0, ..., 0, x_i, 0, ..., 0; z_i) \in h_i(X_i) \subset \boldsymbol{E}_+ = \prod_{i \in I} R_+^{K_i} \times \prod_{j \in J} R_+^{K_j} \times R_+^{K_0} \times R_+^L, \tag{10}$$

where the entry for x_i is the coordinate corresponding to $R_+^{K_i}$. We define $\hat{X}_i$ as the following canonical extension of the range, $h_i(X_i)$, to the open subset of $\boldsymbol{E}_+$, such that

$$\hat{X}_i = h_i(X_i) + R_+^K \times \cdots \times R_+^K \times \{0\} \times R_+^K \times \cdots \times R_+^K \times R_+^K \times \{0\}, \tag{11}$$

where the two entries of $\{0\}$ are representing for those of $R_+^{K_i}$ and R_+^L. Each preference, $\succsim_i$, can naturally be extended to $\hat{\succsim}_i$ on $\hat{X}_i$ by defining it as the relation depending only on the range of h_i. Then, the upper contour set at $h_i(x_i; z_i)$ for $\hat{\succsim}_i$, $\hat{X}_i^{h_i(x_i;z_i)} = \{w \in \hat{X}_i \mid w \hat{\succsim}_i h_i(x_i; z_i)\}$, is nothing but the above canonical extension of image $h_i(X_i^{(x_i;z_i)})$ of the upper contour set at $(x_i; z_i)$ for $\succsim_i$.

Note that for each point $(x^*; y^*) \in Y$ such that $x^* = \sum_{j \in J \cup \{0\}} x_j^{*+}$ and $y^* = \sum_{j \in J \cup \{0\}} y_j^*$, where $(x^*, x_j^{*+}; y_j^*) \in Y_j$ for each $j \in J \cup \{0\}$, if we identify $Y \subset R^{K \cup L}$ with its image in $\boldsymbol{E}$ under the mapping,

$$h : Y \ni (x^*; y^*) \mapsto (x^*, ..., x^*, 0, ..., 0; y^*) \in h(Y) \subset \prod_{i \in I} R^{K_i} \times \prod_{j \in J} R^{K_j} \times R^{K_0} \times R^L, \tag{12}$$

where the entries for x^* are coordinates corresponding to $\prod_{i \in I} R^{K_i}$, the point in $h(Y)$ can be represented as the sum of the canonically identified points of $\hat{Y}_j$, $j \in J \cup \{0\}$, like

$$(x^*, ..., x^*, 0, ..., 0; y^*) = \sum_{j \in J \cup \{0\}} (x_j^{*+}, ..., x_j^{*+}, x_j^{*+} - x_j^{*-}, x_j^{*+}, ..., x_j^{*+}; y_j^*), \tag{13}$$

where the entry for $x_j^{*+} - x_j^{*-}$ for each $j \in J \cup \{0\}$ is the coordinate corresponding to R^{K_j}. Then, by defining $\hat{Y}$ as $\hat{Y} = \sum_{j \in J \cup \{0\}} \hat{Y}_j$, we may identify Y with a subset of $\hat{Y}$ canonically through h as

$$h(Y) \ni h(x^*; y^*) = \sum_{j \in J \cup \{0\}} h_j(x_j^{*-}, x_j^{*+}, y_j^*) \in \sum_{j \in J \cup \{0\}} \hat{Y}_j = \hat{Y}. \tag{14}$$

Moreover, note that a list of Lindahl prices, $((p_I^{i*})_{i \in I}, (p_J^{j*})_{j \in J}, p^{0*}, p_L^*)$, can be identified with a price system over $\boldsymbol{E}$. For each $j \in J \cup \{0\}$, the value of action $(x_j^-, x_j^+, y_j) \in Y_j$ of firm j under the Lindahl price system, $p_K^* \cdot x_j^+ - p_J^{j*} \cdot x_j^- + p_L^* \cdot y_j$, can be represented as

$$((p_I^{i*})_{i \in I}, (p_J^{j*})_{j \in J}, p^{0*}, p_L^*) \cdot h_j(x_j^-, x_j^+, y_j), \text{ for } h_j(x_j^-, x_j^+, y_j) \in \hat{Y}_j, \tag{15}$$

because Lindahl price system always satisfies $p_K^* = \sum_{i \in I} p_I^{i*} + \sum_{j \in J} p_J^{j*} + p^{0*}$. For each $i \in I$, the value of action $(x; z_i) \in X_i$, $p_I^{i*} \cdot x - p_L^* \cdot z_i$, can also be represented as

$$((p_I^{i*})_{i \in I}, (p_J^{j*})_{j \in J}, p^{0*}, p_L^*) \cdot h_i(x, z_i) \text{ for } h_i(x, z_i) \in \hat{X}_i. \tag{16}$$

Hence, conditions (i),(ii) and (iii) of Lindahl equilibrium can be assured through the extended actions in $\hat{Y}_j$ and $\hat{X}_i$ for each $j \in J \cup \{0\}$ and $i \in I$ with the price $((p_I^{i*})_{i \in I}, (p_J^{j*})_{j \in J}, p^{0*}, p_L^*)$ over the extended commodity space, $\boldsymbol{E}$.

Now we can prove the second fundamental theorem. With respect to the allocation, $(x^*; z_i^*)_{i\in I}$, feasible under $(x^*; y^*) \in Y$ with $y^* = y_0^* + \sum_{j\in J} y_j^*$ for $\boldsymbol{\mathcal{E}} = \big((\succsim_i, \omega_i)_{i\in I}, Y_0, (Y_j)_{j\in J}, ((\theta_{ij})_{j\in J})_{i\in I}\big)$, we suppose, at first, that the action of public firm, $(x_0^{*-}, x_0^{*+}; y_0^*)$, is the only element of Y_0 and transform economy $\boldsymbol{\mathcal{E}}$ on $R^{K\cup L}$ into economy $\hat{\boldsymbol{\mathcal{E}}}^*$ on $\boldsymbol{E}$. Define Y_0^* as $Y_0^* = \{(x_0^{*-}, x_0^{*+}; y_0^*)\}$ and Y_j^* as Y_j for each $j \in J$, and consider economy $\boldsymbol{\mathcal{E}}^* = \big((\succsim_i, \omega_i)_{i\in I}, Y_0^*, (Y_j^*)_{j\in J}, ((\theta_{ij})_{j\in J})_{i\in I}\big)$ and its transformation, $\hat{\boldsymbol{\mathcal{E}}}^*$ $= \big((\hat{\succsim}_i, h_i(\omega_i))_{i\in I}, \hat{Y}_0^*, (\hat{Y}_j^*)_{j\in J}, ((\theta_{ij})_{j\in J})_{i\in I}\big)$. Even though Y_0^* is convex, it is not unclear that the set,

$$Y^* = \{(x; y) \mid x = \sum_{j\in J\cup\{0\}} x_j^+, y = \sum_{j\in J\cup\{0\}} y_j, (x, x_j^+; y_j) \in Y_j^* \text{ for all } j \in J \cup \{0\}\}, \tag{17}$$

is convex. However, set $\hat{Y}^* = \sum_{j\in J\cup\{0\}} \hat{Y}_j^* \supset h(Y^*)$ is obviously convex as the sum of convex sets in $\boldsymbol{E}$. Hence, as in Debreu (1959)[Ch.6, 6.4], by considering the class of upper contour sets, $\hat{X}_i^{h_i(x^*; z_i^*)}$, $i \in I$, by incorporating feasibility equation (2), we obtain a supporting price, $((p_I^{i*})_{i\in I}, (p_J^{j*})_{j\in J}, p^{0*}, p_L^*)$, at 0 of the convex set, $-\hat{Y}^* - \sum_{i\in I} h_i(\omega_i) + \sum_{i\in I} \hat{X}_i^{h_i(x^*; z_i^*)}$ in $\boldsymbol{E}$.

It is clear under price represented by $(p_j^{j*})_{j\in J}$ and p_L^*, condition (i) If Lindahl equilibrium is satisfied for all $j \in J$ since every $\hat{Y}_j^*$ is convex. For condition (ii), note that the status quo allocation is Pareto optimal, so there is no allocation that is Pareto superior to it, hence the condition is automatically satisfied. For condition (iii), take $d_i^* \in R$ appropriately so as to satisfy $p_I^{i*} \cdot x^* + p_L^* \cdot z_i^* = p_L^* \cdot \omega_i + \sum_{j\in J} \theta_{ij}\pi_j^* + d_i^*$ for every $i \in I$. ■

4 Conclusion

In this paper, fundamental theorems of welfare economics for economies with public goods are given, including cases in which a Pareto-optimal Lindahl equilibrium allocation is possible only through an inevitable government budget deficit. In particular, (1) medical cares as *merit goods* give concrete significance to the fact that the fundamental theorems of welfare economics may be asserted on the basis of Lindahl equilibrium. (2) The possibility of *deficit financing* is a general equilibrium theoretical confirmation that the private sector does not necessarily share all the costs of public goods, and thus what is *unnecessary tax collection*. (3) With regard to tax collection, our results show the necessity of financing that cannot be realized by the private price in Lindahl equilibria (the undistributed profits means the necessity of *lump-sum taxes for private firms*). Furthermore, (4) in addition, the *activity criteria of the public firm* is given as a kind of maximality of profit, i.e., further Pareto improvement actions inevitably lead to lower profits. (5) Compared to the cost-sharing equilibrium in Mas-Colell and Silvestre (1989), here the question of "what is true cost?" is re-examined through the separation of public and private firms. Savings in the private sector are the purchase of tomorrow 's fiat money, and it is appropriate from the standpoint of general equilibrium theory to treat the amount of such savings as *value creation* by the state. Thus, in an economy with the concept of saturation or savings (i.e., capital accumulation), this amount must be *subtracted from the cost of public good production* as long as we follow the spirit of the Second Fundamental Theorem of Welfare Economics.

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Appendix A

Toward what we described in Example 1 in a static framework, relying for simplicity on 'preference saturation', we would like to add here a supplementary discussion on some dynamic considerations. As mentioned in the explanation of the basic example, the budgetary surplus of a preference-saturated agent in a static dividend equilibrium can be considered as savings, i.e., a loan to the government, but at the same time it must be the final or eternal savings (undistributed profit) under Walras' law. It is the purpose of this appendix to show that such savings can exist even without the preference saturation.

If we allow for 'undistributed profits', directly to the firms, it is not difficult to consider such eternal savings (linked to government budget deficits) in a dynamic framework (think, for example, firms' eternally undistributed internal deposits).[6] In Example 2, we present a dynamic framework in which

[6] Here, we note that the coexistence of undistributed profits with optimality is an interesting feature of the problem concerned when contrasted with the transversality condition in optimal programming theorems for production with discount factors.

similar matters arise without having to consider things like 'undistributed profits'. Under the overlapping generations (consumption loan) framework, the problem can be represented by the *seigniorage* in Samuelson-type savings economies. In the following, we will restate it as a model including production, closer to the basic example in the introduction.

(Example 2): Let us consider a two-period overlapping generations economy such that for each period $t = 1, 2, ...$, there are two commodities (a public good and a private good) and an agent t who lives in periods t and $t+1$ and has a well-behaved technology Y_t that can produce 2 units of private good from one unit of public good in period t under supporting price $(1, 0)$ at action (private, public) $= (2, -1)$. Suppose that public-firm technology Y_0 is such that by using one unit of private good in period 1, it can provide one unit of public good for each period $t = 1, 2, \ldots$ (or does nothing). Moreover, suppose that each agent t has no initial endowment and has a utility function depending only on the amounts of private goods in R^2_+ as $u(x^y, x^o) = x^y + x^o$, where superscripts denote young $= y$ or old $= o$ period for private goods.

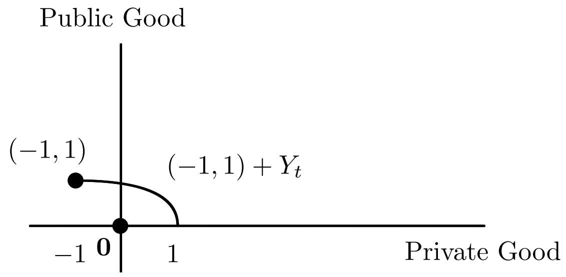

Figure 2: $Y_0 = \{(0, 0), (-1, 1)\}$ and $(-1, 1) + Y_t$

In this case, we also have a trivial (unique) public firm action that is Pareto optimal, "using one unit of private good in period 1 and provide 1 unit of public good for every periods." If otherwise, we have $(0, 0), (0, 0), ...$, as the only feasible allocation. As long as the public firm chooses the action, we have a (unique) Pareto optimal (Lindahl) general equilibrium allocation, $(1, 1), (1, 1), ...$, under price system, $(0, 0, ...)$ for public goods and $(1, 1, ...)$ for private goods with dividend $d_t = 2$ for each $t = 1, 2, ...$ that is equal to the profit for technology Y_t under price (private, public) $= (1, 0)$. For this Lindahl general equilibrium, the profit of public firm is $-1 = (-1) \cdot 1 + 1 \cdot 0 + 1 \cdot 0 + \cdots$, so that the unique Pareto optimal allocation can only be achieved with an inevitable government budget deficits.