Discussion Papers In Economics And Business

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Discussion Paper 26-05

April 2026

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Optimal Supermajority Threshold for Suspending Fiscal Rules*

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April 8, 2026

Abstract

This paper analyzes the optimal supermajority threshold for approving fiscal rule suspensions within a two-period political turnover model. Faced with the potential loss of power, the incumbent party aims to secure preferred expenditures by increasing public debt. To counteract this, an expenditure rule requiring legislative approval for suspension is introduced. The analysis shows that the voting threshold in parliament should exceed a simple majority, making a simple majority rule suboptimal. A stricter supermajority is necessary when fiscal expenditure rules are more flexible, as it allows for more effective responses to economic fluctuations. Moreover, while higher initial debt levels call for stricter expenditure rules, the optimal supermajority threshold remains unaffected by the debt level. Finally, as political polarization among voters intensifies, the optimal threshold decreases, increasingly aligning with the incumbent party's preferences.

Key words: Fiscal rules, Government debt, Political turnover.

JEL Classification: D72, D78, H62, H63

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*The authors would like to thank an anonymous referee for valuable comments and suggestions, which have greatly improved the paper. Financial support from the Japan Society for the Promotion of Science Grants-in-Aid for Scientific Research (C) (No. 25K05011, Ryo Arawatari; No. 24K04926, Tetsuo Ono) is gratefully acknowledged. The authors have no relevant financial or non-financial interests to disclose.

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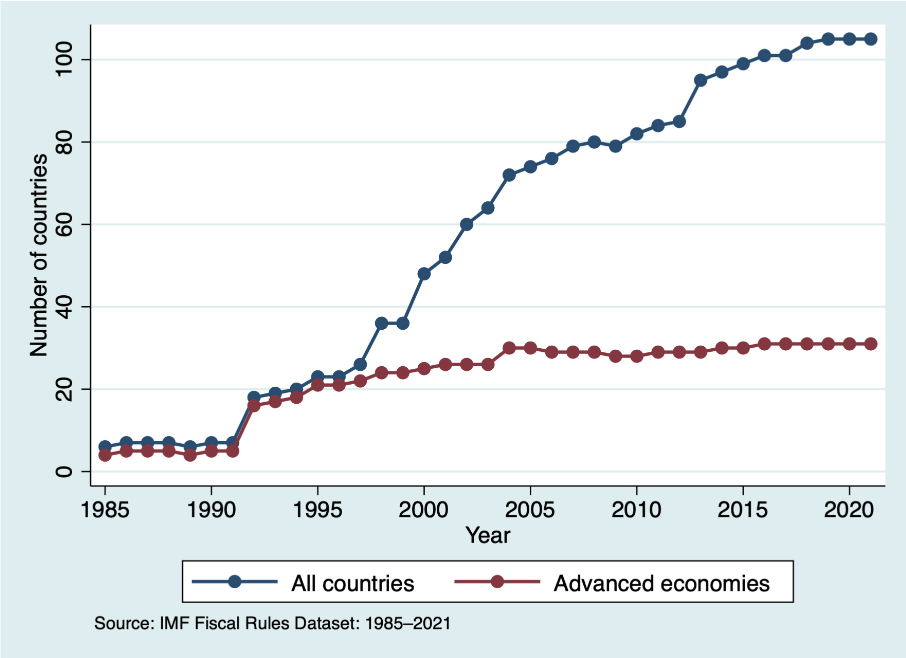

Figure 1: Trends in the number of countries adopting fiscal rules.

1 Introduction

Since the start of the twenty-first century, addressing excessive budget deficits and the continued rise in government debt has emerged as a common policy challenge in advanced and developing countries. To address it, an increasing number of countries have implemented fiscal rules (see Figure 1). Yet, empirical evidence indicates that these rules' adoption has not effectively restrained excessive budget deficit and government debt accumulation (Eyraud et al., 2018). Davoodi et al. (2022) estimate that the likelihood of compliance with fiscal rules among countries adopting them between 2004 and 2021 is approximately 50%. One contributing factor to this ineffective control is the suspension or relaxation of fiscal rules. Fiscal rules are established by laws or constitutions, but they can often be temporarily suspended or exempted from their applications.

Notable evidence of the suspension of fiscal rules is particularly prominent in Japan, which currently holds the highest debt-to-GDP ratio in the world. Japan's Public Finance Act explicitly prohibits the issuance of deficit-financing bonds, with the only exception being the issuance of construction bonds that are strictly earmarked for specific purposes such as public works, capital subscriptions, or lending activities (Financial Bureau of Ministry of Finance, 2019). Nevertheless, the government can bypass these restrictions by enacting a special law known as the Special Deficit Financing Act, which effectively allows for a temporary suspension of the provisions laid out in the Public Finance Act. This special act can be passed relatively easily, requiring only a simple majority in the legislature. Consequently, the ruling party, which often holds the majority, can introduce and pass the act at almost any given time. Over the years, this law has been repeatedly enacted, resulting in the frequent issuance of deficit-financing bonds. As a result, the original intent and restrictions of the Public Finance Act have been largely circumvented, rendering it ineffective in practice.

The suspension or relaxation of fiscal rules through majority voting approval is also observed in both Germany and the United States. In Germany, the national government's budget must, in principle, be balanced without resorting to borrowing. However, an emergency exception can be invoked with the consent of an absolute majority in the national parliament (Mause and Groeteke, 2012). In the United States, a filibuster allows senators to extend debate indefinitely, effectively blocking a vote on legislation. Ending a filibuster requires a cloture process, which needs the support of three-fifths of the Senate to proceed to a vote (Congressional Research Service, 2017). Nevertheless, fiscal reconciliation, a legislative process that requires only a simple majority, can be used to expedite the passage of certain budget-related bills, especially those related to federal revenues, expenditures, and the debt ceiling (Congressional Research Service, 2023). Thus, simple majority approval for suspending fiscal rules is a prevalent trend in developed countries.

The suspension of fiscal rules shapes fiscal policy and annual budget decisions, which, in turn, impact household utility. Consequently, whether the government should have the option to suspend fiscal rules or be constitutionally prohibited from doing so has significant implications for social welfare. Additionally, if the option to suspend fiscal rules is permitted, determining whether the decision should rely on a simple majority vote in the legislature also carries significant implications for social welfare. The aim of this study is to examine the optimality of allowing fiscal rule suspensions and evaluating whether a simple majority vote—a common practice in many developed countries—should underpin such decisions.

For this aim, this study constructs a model that incorporates an expenditure rule (Piguillem and Riboni, 2021) within a two-period framework of political turnover, drawing upon the frameworks proposed by Persson and Svensson (1989) and Alesina and Tabellini (1990). The model considers two distinct categories of public expenditure and accounts for two types of voters with differing public expenditure preferences, each represented by a political party. Voters' preferences for public expenditure are assumed to evolve in response to changes in social and economic contexts. Consequently, each party faces the risk of alternation in power due to stochastic shifts in voter composition.

Confronted with the prospect of political turnover, the incumbent party is motivated to augment its current preferred public expenditure through increased public debt issuance. This strategy stems from the concern that its favored public expenditure may face reductions should it lose power in the future. Consequently, political disputes over public expenditure and the potential for changes in power lead to excessive budget deficits. To address these deficits, we supplement the standard model of political turnover with two types of rules: an expenditure rule, which imposes a cap on public expenditure, and a supermajority threshold, which outlines conditions for suspension of the expenditure rule. Specifically, the study examines an environment where the suspension of the expenditure rule is permissible if the supermajority, rather than the simple majority in legislature members approve it. This feature distinguishes the model used here.

There are two views of justifications for suspending fiscal rules. The first concerns extreme adverse shocks, such as a pandemic or a severe financial crisis, where fiscal rules may excessively constrain a government's ability to respond appropriately. The second concerns changes in society's preferences: when voters grant a government a strong electoral mandate, fiscal rules may be seen as an obstacle to implementing policies that reflect the will of the people. This study focuses on the trade-off arising from the second view—how to control politicians' spending bias while simultaneously endowing them with enough flexibility to respond to society's changing demands for public expenditure. By framing the problem in this way, we highlight that the role of the supermajority threshold is not merely to provide an escape valve for emergencies, but rather to serve as a mechanism that grants greater fiscal discretion to governments that enjoy broader popular support.

The trade-off underlying our analysis is embedded in the social welfare function defined in the following way. In our model, the proportion of legislative seats held by the incumbent party, p, also reflects the proportion of voters who prefer the incumbent's favored public expenditure. Adopting a Benthamite criterion, the social welfare function, W, weights each party's value by the number of its supporters, yielding $W = \int_0^1 [p\, V_A + (1-p)\, V_B]\, dp$, where V_A and V_B denote the value functions for parties A and B, representing voters of type A and B, respectively. Because a party holds office only when it commands at least half of the seats ($p \geq 1/2$), this welfare function assigns a higher weight to the incumbent's welfare: the larger the incumbent's seat share, the greater the weight the social planner places on its preferences. The interpretation is that the legislature's composition represents a genuine shift in society's preferences for public goods, and the planner accordingly wishes to endow governments with broader popular support with greater fiscal discretion.

Ideally, the social planner would choose a fiscal rule contingent on the realization of p, granting more spending flexibility to governments with larger seat shares. However, in practice, fiscal rules and supermajority thresholds are typically enshrined in constitutions or laws before elections take place and cannot be adjusted to each electoral outcome. Given this constraint, the supermajority threshold δ plays a key role as a second-best instrument. The planner chooses δ such that governments with $p \geq \delta$ are free to determine fiscal policy without the expenditure rule, while governments with $p < \delta$ must abide by it.

Thus, the supermajority threshold balances two objectives: imposing fiscal discipline on governments with moderate popular support, while allowing governments with strong mandates to respond flexibly to the public expenditure demands of their constituents. This welfare-based rationale for the supermajority threshold distinguishes our approach from that of Piguillem and Riboni (2021), who assign equal weights to the two parties. While equal weights may be appropriate in a stationary environment, our formulation better captures the idea that in a changing world, a government's fiscal discretion should reflect the degree of popular support it commands.

In light of the properties of the social welfare function outlined above, we examine the optimal supermajority threshold, δ, representing the minimum required proportion of votes necessary to approve the suspension of the expenditure rule in the legislature, from three perspectives. First, we analyze the optimal combination of the expenditure rule and the supermajority threshold, focusing on maximizing social welfare. The results indicate that, under this optimal combination, the supermajority threshold, δ, is less than unanimity, implying that the optimal combination does not entirely prohibit the suspension of the expenditure rule. Additionally, δ exceeds $1/2$, suggesting that a simple majority threshold (i.e., $\delta = 1/2$) is suboptimal for social welfare in suspension decisions. Furthermore, a stricter supermajority threshold correlates with a more relaxed expenditure rule. Thus, when fiscal expenditure is subject to fewer constraints under the expenditure rule, the supermajority threshold should be set higher to allow governments with strong popular support to respond to shifts in society's preferences.

Second, we clarify the relationship between the fiscal situation and the optimal supermajority threshold. Specifically, we analyze the optimal supermajority threshold's characteristics given the initial debt balance. Under the CRRA preferences adopted in this paper, we find that as the initial debt balance increases, a stricter expenditure rule becomes optimal, whereas the optimal supermajority threshold remains unaffected by the initial debt balance. The intuition behind this result is as follows: The supermajority threshold governs the balance between fiscal flexibility and the cost of politically biased expenditure. A change in the initial debt level alters the total fiscal resources available, but it affects both sides of this trade-off proportionally, leaving the optimal balance point unchanged. Consequently, we suggest that the supermajority threshold should maintain consistency over the long-term following the economic structure, rather than changing it in response to changes in the fiscal situation.

Third, we clarify the relationship between the magnitude of political conflict among voters (i.e., between parties) and the optimal supermajority threshold. Specifically, we investigate how differences in preferences for public goods expenditure among voters impact the optimal supermajority threshold. We find that as political conflict among voters increases, the optimal supermajority threshold decreases. Given more support of the incumbent party than the opposition party, the optimal supermajority threshold aligns more with the incumbent's preferences as political conflict evolves. Consequently, a larger disparity in preferences leads to a preference for more lenient supermajority and expenditure rules.

The remainder of this article is organized as follows. Section 2 reviews the relevant literature. Section 3 presents the model. Section 4 examines the optimal combination of an expenditure rule and a supermajority threshold. Section 5 conducts a comparative statics analysis to explore the effects of initial public debt levels and variations in preferences between parties on the optimal fiscal threshold. Section 6 provides concluding remarks. The appendix provides the proofs of the lemmas and propositions.

2 Related Literature

This study is closely related to Piguillem and Riboni (2021), who developed a strategic debt model in which preference misalignment between current and future governments creates incentives for the incumbent to overissue public debt (Persson and Svensson, 1989; Alesina and Tabellini, 1990). The authors introduce the possibility for politicians to circumvent fiscal rules with a consensus between the incumbent and the opposition. Within this framework, they examine which fiscal rule is the most effective in promoting interparty compromise and reducing debt. We adopt their framework but determine the threshold required in a supermajority vote to approve exemptions from the applications of fiscal rules rather than allowing these exemptions to be negotiated between parties. In this alternative framework, we characterize the optimal combination of an expenditure rule and a supermajority threshold for allowing the rule suspension through voting from the perspective of maximizing social welfare. Our findings provide a basis for evaluating the simple majority rule for fiscal rule suspensions, currently or formerly implemented in countries such as Japan and Germany, from a social welfare standpoint.

This study also relates to the political economy literature on fiscal rule circumvention (Coate and Milton, 2019; Dovis and Kirpalani, 2020; Halac and Yared, 2022; Arawatari and Ono, 2021, 2022). In Coate and Milton's (2019) analysis, two pivotal roles emerge: the constitutional designer, representing citizens, and the politician, who is inclined towards higher taxes. The designer establishes a fiscal limit to curb the politician's bias, yet the politician can override it with citizen approval. The authors focus on determining the optimal fiscal limit and examining its sensitivity to potential overrides. However, the static analytical framework excludes expenditure financing through bond issues and the resulting budget deficits. Consequently, fiscal rules controlling deficits are beyond the study's scope.

Dovis and Kirpalani (2020) present various instances in which fiscal rules were established but subsequently lacked proper enforcement, as seen in the case of the Stability and Growth Pact within the European Union (EU). Their analysis introduces a model of local and central governments to explain these situations. In their framework, local governments can override fiscal rules, leading to excessive borrowing; however, the authors do not address how optimal fiscal rules should be determined for these specific cases.

Halac and Yared (2022) examine a government displaying a present bias towards public expenditure while possessing private information about shocks impacting the value of such expenditure. Society selects a fiscal rule, aiming to balance the advantages of committing the government to avoid over-expenditure against the advantages of allowing flexibility to respond to shocks. The authors demonstrate that rule violation occurs under sufficiently high shocks only when the penalties are weak and such shocks are relatively unlikely. However, the authors do not explore the optimal design of fiscal rules in the context of potential fiscal rule violations.

Arawatari and Ono (2021, 2022) also examine a present-biased government's behavior, as in Halac and Yared (2022). Arawatari and Ono (2021, 2022) utilize the model of Bisin et al. (2015), which incorporates time-inconsistent voters inclined to increase present consumption by raising public debt issuance. However, imposing a debt ceiling can reduce the influence of this present-biased behavior. Arawatari and Ono (2021, 2022) extend this analysis by allowing the debt ceiling to be circumvented. They illustrate instances of such circumvention (Arawatari and Ono, 2021) and present an optimal debt rule under the possibility of circumvention (Arawatari and Ono, 2022). However, they do not address voting for allowing exemptions from applying the debt rule. In contrast, the present study demonstrates such a rule within a political economy framework, a facet that has not been fully covered in the literature.

3 The Model

Our model is based on a standard two-period framework that incorporates political turnover (Persson and Svensson, 1989; Alesina and Tabellini, 1990). The economic setting involves two distinct public expenditure variables, denoted as g^A and g^B, for two categories of voters, A and B, respectively, who have disparate public expenditure preferences. Each voter type's preferences in each period are specified as follows:

$$u_A(g_A, g_B) = \frac{(g_A)^{1-\sigma}}{1-\sigma} + \theta \cdot \frac{(g_B)^{1-\sigma}}{1-\sigma}, \tag{1}$$

$$u_B(g_A, g_B) = \frac{(g_B)^{1-\sigma}}{1-\sigma} + \theta \cdot \frac{(g_A)^{1-\sigma}}{1-\sigma}, \tag{2}$$

where $\sigma \in (0, 1)$ and $\theta \in [0, 1)$ are parameters shaping the preferences of each type of voters. The functions $u_A(g_A, g_B)$ and $u_B(g_A, g_B)$ represent the preferences of voters favoring public expenditure g^A and g^B, respectively. A smaller value of θ intensifies the political conflict surrounding public expenditure.

The size of each voter type is stochastically determined in each period. This stochastic process is the only source of uncertainty in the economy and can be interpreted as a form of preference shock, reflecting changes in voters' preferences for public expenditure in response to shifts in social and economic environments. We consider two political parties exist, A and B, representing voters of types A and B, respectively. Given their symmetric preferences, we denote the incumbent's and opposition's current preferences in each period as follows:

$$u_I(g_I, g_O) = \frac{(g_I)^{1-\sigma}}{1-\sigma} + \theta \cdot \frac{(g_O)^{1-\sigma}}{1-\sigma}, \tag{3}$$

$$u_O(g_I, g_O) = \frac{(g_O)^{1-\sigma}}{1-\sigma} + \theta \cdot \frac{(g_I)^{1-\sigma}}{1-\sigma}, \tag{4}$$

where the indices I and O signify incumbent and opposition, respectively.

Assuming that public expenditure, denoted as $\{g_I, g_O\}$, is funded through tax revenue and public debt issuances, the government's budget constraint for each period is expressed as follows:

$$\tau + b' \geq (1+r)b + g_I + g_O, \tag{5}$$

where $\tau > 0$ represents exogenous tax revenue in each period, b represents the outstanding public debt at the beginning of the period, b' denotes the amount of public debt issued during the period, and r is the interest rate on public debt. We assume that tax revenue remains constant to focus on the conflict between the two political parties regarding allocating two distinct public expenditures. We also assume that the country is a small open economy, borrowing within foreign asset markets at a given interest rate r.

With $\theta < 1$ in (3) and (4), the incumbent party faces a reduced incentive to save resources for the second period, since with some probability it will lose power and the opposition will then allocate the remaining public budget to a public goods that the incumbent dislikes. Anticipating this possibility, the incumbent increases current preferred public expenditure g_I by issuing additional public debt b'. In essence, political conflict over the composition of public expenditure, together with the prospect of political turnover, contributes to the emergence of excessive budget deficits.

Political turnover uncertainty arises because of the stochastic variation in the number of voters of each type. Let p denote the number of seats obtained by the incumbent in the previous period through elections in the current period: it is assumed to follow a uniform distribution within the interval $[0, 1]$. The party that secures the majority of seats holds power. Under these assumptions, the probability of political turnover in each period is 1/2.

3.1 Expenditure Rule and Supermajority Threshold

We examine the expenditure rule and the supermajority threshold required to approve an exemption from it. To emphasize the role of the expenditure rule, we focus on discretionary public expenditure, as outlined by Piguillem and Riboni (2021), and specify the rule as follows:

$$g_I + g_O \leq \alpha \cdot (\tau - rb), \quad \alpha \geq 0, \tag{6}$$

where $\alpha \geq 0$ serves as a parameter showing the expenditure rule's stringency. The expenditure rule in (6) dictates that the party in office is restricted to allocate only a certain percentage of tax revenue, net of interest payments on debt. A smaller value of α corresponds to a more rigorous expenditure rule. Specifically, $\alpha = 0$ denotes a shutdown rule, entailing no discretionary expenditure ($g_I + g_O = 0$). Conversely, $\alpha = 1$ denotes adherence to a balanced budget rule, resulting in no public debt balance change ($g_I + g_O + rb = \tau$). When $\alpha = \tau/(\tau - rb)$, a primary balance equilibrium rule exists, ensuring that public expenditure is entirely covered by tax revenues ($g_I + g_O = \tau$).

The key distinction in our model, which differentiates it from that of Piguillem and Riboni (2021), is the explicit consideration of a supermajority threshold for approving the suspension of fiscal rules. We assume that if a political party wins an election and obtains the proportion of seats above $\delta \in [1/2, 1]$, it is entitled to implement fiscal policies without being constrained by the expenditure rule. The parameter δ is an exogenously assigned institutional parameter defining the supermajority threshold.

Building upon Piguillem and Riboni (2021), we designate the scenario wherein the incumbent secures more (less) than δ seats as the "dictator (rule) state." Figure 2 illustrates the relationship between the number of seats p held by party A, which held power in the previous period, and applicable expenditure rule for the party in office in the current period. If party A continues to retain power in the current period, it must adhere to the expenditure rule when the

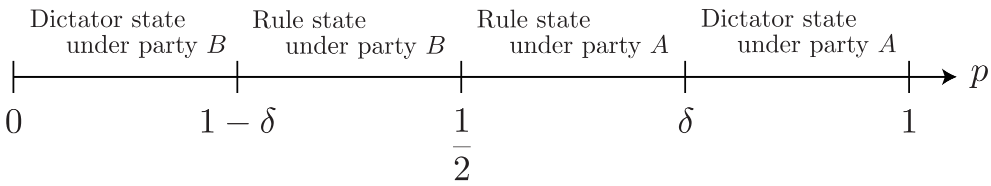

Figure 2: Relationship between the random variable p and incumbent party in the current period if the incumbent in the previous period was party A.

probabilistically determined number of seats p falls within the range $1/2 \leq p < \delta$ (rule state). Conversely, if $\delta \leq p$, party A may invoke the supermajority rule, bypassing the expenditure rule (dictator state). The equilibrium fiscal policy in the dictator state is denoted as $\{g_I^{d*}, g_O^{d*}, b'^{d*}\}$, while the equilibrium fiscal policy in the rule state is represented as $\{g_I^{r*}, g_O^{r*}, b'^{r*}\}$.

When $\delta = 1/2$, the ruling party's number of seats will always exceed the supermajority threshold, δ. Thus, it consistently operates without the expenditure rule, akin to the standard regime change model without fiscal constraints (Alesina and Tabellini, 1990). Conversely, when $\delta = 1$, it is obligated to consistently adhere to the expenditure rule, aligning with the regime change model commonly employed in prior studies on fiscal rules (Piguillem and Riboni, 2021). By specifically examining cases where $1/2 < \delta < 1$, we analyze the supermajority threshold's impact on equilibrium fiscal policy, and explore the optimal interplay between the expenditure rule and the supermajority thresholds.

3.2 Timing of Events

Policy decisions in a two-period economy begin as follows. In the first period (period 1), the random variable $p \in [0, 1]$ determines how seats are divided between the political parties A and B. The party with more than half of the seats runs for office, with the authority to shape fiscal policies, including public goods provision and public debt issuance for given tax revenue, τ. In the rule state where p falls within the range $(1-\delta, \delta)$, the party in the office creates and enacts fiscal policies following the expenditure rule. However, in the dictator state where p falls within the range $[\delta, 1]$ or $[0, 1-\delta]$, the party in the office can decide on fiscal policies without adhering to the expenditure rule.

In the second period (period 2), a new realization of the random variable $p \in [0, 1]$ determines if the incumbent party from the previous period still stays in office. Recall that we assume a uniform distribution for the number of seats p. The incumbent party from period 1 may become the opposition in period 2. Consequently, in period 1, it decides fiscal policies considering the possibility. We address the period-2 problem before examining the period-1 problem.

3.3 Period-2 Incumbent Party's Problem

In period 2, the incumbent party has no opportunity to issue additional debt because the economy ends at the end of period 2. Furthermore, the incumbent party in period 2 is not subject to the expenditure rule. The rationale of this assumption is as follows. If the expenditure rule were binding in period 2, the government would be compelled to generate a fiscal surplus. However, since the economy terminates at the end of period 2, maintaining a surplus would yield no benefit to either the incumbent or the opposition. Consequently, in period 2, both parties (at least in a weak sense) agree to disregard the expenditure rule. As a result, the incumbent party in period 2 can determine fiscal policy without being constrained by the expenditure rule, irrespective of its number of seats in the legislature. Accordingly, the period-2 incumbent party faces the following problem:

$$V_{I,2} = \max_{\{g_{I,2}, g_{O,2}\}} \{u_I(g_{I,2}, g_{O,2})\}, \tag{7}$$

$$\text{s.t. } \tau \geq (1+r)b_1 + g_{I,2} + g_{O,2}, \tag{8}$$

where $V_{I,2}(b_1)$ is the value function of the incumbent party in period 2.

The supply of public goods chosen by the incumbent party in period 2, $\{g^*_{I,2}, g^*_{O,2}\}$, is given by

$$g^*_{I,2}(b_1) = \frac{\tau - (1+r)b_1}{1+\theta^{\frac{1}{\sigma}}}, \quad g^*_{O,2}(b_1) = \theta^{\frac{1}{\sigma}} \cdot \frac{\tau - (1+r)b_1}{1+\theta^{\frac{1}{\sigma}}}, \tag{9}$$

where $g^*_{I,2}(b_1) > 0$ and $g^*_{O,2}(b_1) > 0$ hold under a plausible assumption, which will be verified in Section 3.4. Equation (9) implies that a smaller θ leads to a smaller ratio $g^*_{O,2}(b_1)/g^*_{I,2}(b_1)$. In other words, the stronger the political conflict (smaller θ), the lower the supply of the opposition's preferred public good $g_{O,2}$. Equation (9) further indicates that a larger public debt outstanding in period 2, b_1, reduces the supply of public goods. Accordingly, the incumbent party in period 1 can affect the fiscal policy of the party in office in period 2 through the public debt, b_1, inherited by the latter.

By substituting equation (9) into equation (7), we obtain the value function of the incumbent party in period 2 as follows:

$$V_{I,2}(b_1) = u_I\left(g^*_{I,2}(b_1), g^*_{O,2}(b_1)\right) = \frac{1+\theta^{\frac{1}{\sigma}}}{1-\sigma} \cdot \left(\frac{\tau-(1+r)b_1}{1+\theta^{\frac{1}{\sigma}}}\right)^{1-\sigma}. \tag{10}$$

The value function of the opposition party in period 2 is:

$$V_{O,2}(b_1) = u_O\left(g^*_{I,2}(b_1), g^*_{O,2}(b_1)\right) = \frac{\theta+\theta^{\frac{1-\sigma}{\sigma}}}{1-\sigma} \cdot \left(\frac{\tau-(1+r)b_1}{1+\theta^{\frac{1}{\sigma}}}\right)^{1-\sigma}. \tag{11}$$

Equations (10) and (11) imply that the ratio of the incumbent party's value to that of the opposition party is given by

$$\frac{V_{I,2}(b_1)}{V_{O,2}(b_1)} = \frac{1+\theta^{\frac{1}{\sigma}}}{\theta+\theta^{\frac{1-\sigma}{\sigma}}} = 1 + \frac{(1-\theta)\left(1-\theta^{\frac{1-\sigma}{\sigma}}\right)}{\theta+\theta^{\frac{1-\sigma}{\sigma}}} > 1. \tag{12}$$

Therefore, $V_{I,2}(b_1) > V_{O,2}(b_1)$ holds, indicating that the incumbent party's value exceeds that of the opposition party as a consequence of political conflict.

3.4 Period-1 Incumbent Party's Problem

Next, we consider the period-1 incumbent party's problem. Public debt from the outset of the period 1, denoted as b_0, is a predetermined variable, and thus, an initial condition for the incumbent party. For this value of b_0, we assume:

Assumption 1 $b_0 < \dfrac{2+r}{(1+r)^2} \cdot \tau \equiv b^{NDL}$.

Note that b^{NDL} denotes the natural debt ceiling.[1] The problem confronting the incumbent party in period 1 hinges on whether its number of seats exceeds δ (dictator state) or falls below δ (rule state). We investigate the period-1 incumbent party's decision-making process for both the dictator and rule state scenarios.

[1] The natural debt ceiling is derived as follows. The government's total debt repayment obligation in period 1 amounts to $(1+r)b_0$, which must not exceed the present discounted value of tax revenues to ensure repayment. Accordingly, the initial debt burden b_0 must satisfy:

$$(1+r)b_0 \leq \tau + \frac{\tau}{1+r} \;\Leftrightarrow\; b_0 \leq \frac{2+r}{(1+r)^2} \cdot \tau \equiv b^{NDL}.$$

3.4.1 Dictator State Case

Consider the situation where the incumbent party's number of seats in period 1 exceeds the supermajority threshold, δ, thereby realizing the dictator state. Then, the incumbent party possesses the flexibility to implement fiscal policies without being constrained by the expenditure rule. Consequently, the problem faced by the period-1 incumbent party can be defined as follows.

$$V_{I,1}^{d} = \max_{\{g_{I,1}, g_{O,1}, b_1\}} \left\{ u_I(g_{I,1}, g_{O,1}) + \beta W_2(b_1) \right\}, \tag{13}$$

$$\text{s.t. } \tau + b_1 \geq (1+r)b_0 + g_{I,1} + g_{O,1}, \tag{14}$$

where $\beta \in (0, 1)$ is the discount factor and $W_2(b_1)$ is the expected value function of the next period.

The allocation of seats for each party in the subsequent period is determined irrespective of the current seat count. Given the uniform distribution of p, the probabilities of being in and out of office in period 2 are equal at 1/2. The expected value function for the next period, denoted $W_2(b_1)$, remains identical for both the incumbent and opposition parties, and can be expressed as follows:

$$\begin{aligned} W_2(b_1) &= \frac{1}{2} \cdot V_{I,2}(b_1) + \frac{1}{2} \cdot V_{O,2}(b_1) \\ &= \phi \cdot \frac{1}{1-\sigma} \cdot \left(\frac{\tau - (1+r)b_1}{1 + \theta^{\frac{1}{\sigma}}} \right)^{1-\sigma}, \end{aligned} \tag{15}$$

where ϕ is defined as follows:

$$\phi \equiv \frac{1 + \theta^{\frac{1}{\sigma}} + \theta + \theta^{\frac{1-\sigma}{\sigma}}}{2}. \tag{16}$$

To clarify the relationship between expenditure rule and fiscal policy, we introduce the following assumption.

Assumption 2 $\beta(1 + r) = 1$.

This assumption abstracts from the incumbent party's incentive to save and borrow considering intertemporal optimization. That is, the motivation to issue public debt arises solely from the risk of regime change. This assumption is also a prerequisite for steady-state stability in the infinite-horizon small open economy model.

In the dictator state, the set of fiscal policies selected by the incumbent party is represented as $\{b_1^{d*}(b_0), g_{I,1}^{d*}(b_0), g_{O,1}^{d*}(b_0)\}$. The policies are expressed as:

$$b_1^{d*}(b_0) = \frac{\tau - \left(\frac{\phi}{1+\theta^{\frac{1}{\sigma}}}\right)^{\frac{1}{\sigma}} \cdot [\tau - (1+r)b_0]}{(1+r) + \left(\phi/(1+\theta^{\frac{1}{\sigma}})\right)^{\frac{1}{\sigma}}}, \tag{17}$$

$$g_{I,1}^{d*}(b_0) = \frac{\tau + (1+r)\,[\tau - (1+r)b_0]}{(1+\theta^{\frac{1}{\sigma}})\left[(1+r) + \left(\phi/(1+\theta^{\frac{1}{\sigma}})\right)^{\frac{1}{\sigma}}\right]}, \tag{18}$$

$$g_{O,1}^{d*}(b_0) = \theta^{\frac{1}{\sigma}} \cdot \frac{\tau + (1+r)\,[\tau - (1+r)b_0]}{(1+\theta^{\frac{1}{\sigma}})\left[(1+r) + \left(\phi/(1+\theta^{\frac{1}{\sigma}})\right)^{\frac{1}{\sigma}}\right]}, \tag{19}$$

where $g_{I,1}^{d*}(b_0) > 0$ and $g_{O,1}^{d*}(b_0) > 0$ are satisfied under Assumption 1. Moreover, using (17), we can verify that $g_{I,2}^{*}(b_1^{d*}) > 0$ and $g_{O,2}^{*}(b_1^{d*}) > 0$ obtain from equation (9), since $\tau-(1+r)b_1^{d*}(b_0) > 0$ is valid under Assumption 1.

Equations (18) and (19) indicate that government expenditures in the dictatorship state depend on the degree of conflict between the two parties, represented by θ. The following lemma establishes the implication of this relationship.

Lemma 1 *The greater the conflict between the parties (with a smaller θ), the larger the government expenditure $\left(g_{I,1}^{d*}(b_0) + g_{O,1}^{d*}(b_0)\right)$ in the dictator state.*

Proof. See Appendix A.

The result presented in Lemma 1 implies that the risk of political turnover and conflict between parties induces excessive fiscal expenditure, consistent with Alesina and Tabellini (1990, Proposition 5). The mechanism operates as follows. In period 1, the incumbent party faces the possibility of losing power in period 2. Should it become the opposition, the residual public budget will be directed toward public goods that are not aligned with the period-1 incumbent's preferences. Anticipating this, the incumbent has weaker incentives to preserve resources for the second period and instead stronger incentives to channel resources into its own preferred public goods while in office. The lower the value of θ and the sharper the conflict between parties, the stronger these incentives become, resulting in higher public expenditure and greater issuance of public debt.

When no expenditure rule is imposed, public goods provision is determined by equations (18) and (19). With the expenditure rule's introduction, the condition under which the incumbent party is motivated to deviate from the expenditure rule can be expressed as follows.

$$g_{I,1}^{d*}(b_0) + g_{O,1}^{d*}(b_0) > \alpha \cdot (\tau - rb_0) \;\Leftrightarrow\; \frac{\tau + (1+r)\left[\tau - (1+r)b_0\right]}{(\tau - rb_0)\left[(1+r) + \left(\phi/(1+\theta^{\frac{1}{\sigma}})\right)^{\frac{1}{\sigma}}\right]} > \alpha, \tag{20}$$

When the condition in (20) holds, the expenditure rule (6) is binding. Then, the incumbent party in period 1 has an incentive to deviate from the expenditure rule. Put differently, in the rule state, the incumbent party must determine its fiscal policy within the confines of both the expenditure and the supermajority rules. We introduce the following assumption for this case:

Assumption 3 $\alpha < \dfrac{\tau + (1+r)\left[\tau - (1+r)b_0\right]}{(\tau - rb_0)\left[(1+r) + \left(\phi/(1+\theta^{\frac{1}{\sigma}})\right)^{\frac{1}{\sigma}}\right]} \equiv \bar{\alpha}(b_0),$

$\bar{\alpha}(b_0)$ is a decreasing function of both b_0 and θ. The greater the initial public debt balance (b_0) or the lesser the conflict between parties (resulting in a larger θ), the weaker the incentive for the period-1 incumbent party to provide excessive public goods. Consequently, the expenditure rule does not bind unless it is stronger. In addition, as demonstrated later, the optimal expenditure rule that maximizes social welfare consistently adheres to the condition $\alpha^* < \bar{\alpha}(b_0)$.

Derived from equations (13), (15), (17), and $\bar{\alpha}(b_0)$, the value functions for both the incumbent and opposition parties in the dictator state are respectively as follows.

$$V_{I,1}^{d}(b_0) = \frac{1}{1-\sigma} \cdot \left[1 + \theta^{\frac{1}{\sigma}} + \beta(1+\theta^{\frac{1}{\sigma}})\left(\frac{\phi}{1+\theta^{\frac{1}{\sigma}}}\right)^{\frac{1}{\sigma}}\right] \cdot \left[\frac{\bar{\alpha}(b_0)(\tau - rb_0)}{1+\theta^{\frac{1}{\sigma}}}\right]^{1-\sigma}, \tag{21}$$

$$V_{O,1}^{d}(b_0) = \frac{1}{1-\sigma} \cdot \left[\theta + \theta^{\frac{1-\sigma}{\sigma}} + \beta(1+\theta^{\frac{1}{\sigma}})\left(\frac{\phi}{1+\theta^{\frac{1}{\sigma}}}\right)^{\frac{1}{\sigma}}\right] \cdot \left[\frac{\bar{\alpha}(b_0)(\tau - rb_0)}{1+\theta^{\frac{1}{\sigma}}}\right]^{1-\sigma}, \tag{22}$$

where $V_{I,1}^{d}(b_0) > V_{O,1}^{d}(b_0)$ holds from the condition in (12).

3.4.2 Rule State Case

When the incumbent party has fewer seats than the supermajority threshold, denoted as δ, the rule state is established. The incumbent party is obligated to formulate fiscal policy in adherence to the expenditure rule. Consequently, the incumbent party's fiscal problem is as follows:

$$V^r_{I,1} = \max_{\{g_{I,1}, g_{O,1}, b_1\}} \{u_I(g_{I,1}, g_{O,1}) + \beta W_2(b_1)\}, \tag{23}$$

$$\text{s.t. } \tau + b_1 \geq (1+r)b_0 + g_{I,1} + g_{O,1}, \tag{24}$$

$$g_{I,1} + g_{O,1} \leq \alpha(\tau - rb_0). \tag{25}$$

The problem faced by the incumbent party in the rule state differs from that in the dictator state in that it involves adhering to the expenditure rule of the form in (25). Furthermore, under Assumption 3, the expenditure rule holds with equality. For this, expression in (25) must hold with an equality sign in the analysis of the ruling state.

Let $\{b^{r*}_1(b_0, \alpha), g^{r*}_{I,1}(b_0, \alpha), g^{r*}_{O,1}(b_0, \alpha)\}$ denote the fiscal policy chosen by the incumbent party in the rule state. Using $g^{r*}_{O,1} = \theta^{\frac{1}{\sigma}} g^{r*}_{I,1}$ derived from the first-order condition, we ascertain that the pair of public expenditure $\{g^{r*}_{I,1}(b_0, \alpha), g^{r*}_{O,1}(b_0\alpha)\}$ satisfies the following equation:

$$g^{r*}_{I,1} + g^{r*}_{O,1} = (1 + \theta^{\frac{1}{\sigma}})g^{r*}_{I,1} = \alpha(\tau - rb_0). \tag{26}$$

Utilizing the condition in (26) met by the pair of public expenditure and the budget constraint in (24), we derive the fiscal policy in the rule state as follows:

$$b^{r*}_1(b_0, \alpha) = b_0 + (\alpha - 1)(\tau - rb_0), \tag{27}$$

$$g^{r*}_{I,1}(b_0, \alpha) = \frac{\alpha(\tau - rb_0)}{1 + \theta^{\frac{1}{\sigma}}}, \tag{28}$$

$$g^{r*}_{O,1}(b_0, \alpha) = \theta^{\frac{1}{\sigma}} \cdot \frac{\alpha(\tau - rb_0)}{1 + \theta^{\frac{1}{\sigma}}}. \tag{29}$$

where $g^{r*}_{I,1}(b_0, \alpha) > 0$ and $g^{r*}_{O,1}(b_0, \alpha) > 0$ hold under Assumption 1. Moreover, using equation (27), we can verify that $g^*_{I,2}(b^{r*}_1) > 0$ and $g^*_{O,2}(b^{r*}_1) > 0$ hold from equation (9), since $\tau - (1 + r)b^{r*}_1(b_0, \alpha) > 0$ holds under Assumption 1.[2]

[2]From equation (27) and Assumptions 1 and 3, we obtain

$$\begin{aligned}
\tau - (1+r)b^{r*}_1(b_0, \alpha) &> \tau - (1+r)b^{r*}_1(b_0, \bar{\alpha}(b_0)) \\
&= \tau - (1+r)b_0 + (1+r)(\tau - rb_0) - (1+r) \cdot \frac{\tau + (1+r)\left[\tau - (1+r)b_0\right]}{(1+r) + \left(\phi/(1+\theta^{\frac{1}{\sigma}})\right)^{\frac{1}{\sigma}}} \\
&> \tau - (1+r)b_0 + (1+r)(\tau - rb_0) - (1+r) \cdot \frac{\tau + (1+r)\left[\tau - (1+r)b_0\right]}{1+r} \\
&= 0.
\end{aligned}$$

The expressions in equations (27), (28), and (29) indicate that the expenditure rule influences fiscal policy in period-1 of the rule state. Specifically, a stricter expenditure rule (smaller α) leads to more effective control over excessive budget deficits. Essentially, the expenditure rule helps restrain unwarranted fiscal expenditures arising from the risk of political turnover and conflict, as highlighted in Lemma 1. The remainder of this section examines how the impact of the expenditure rule parameter, α, on fiscal policy shapes the difference in preferences toward α between the incumbent and opposition parties. This analysis may be helpful for the optimal rule analysis in the next section.

Substitution of the required fiscal policy, as indicated by equations (27), (28), and (29), in the incumbent party's objective function (23) and expected value of the next period equation (15) yields the following value functions of the incumbent and opposition parties under the rule state:

$$V_{I,1}^{r}(b_0,\alpha)=\frac{1}{1-\sigma}\cdot\left\{\left(1+\theta^{\frac{1}{\sigma}}\right)\cdot\left[\frac{\alpha(\tau-rb_0)}{1+\theta^{\frac{1}{\sigma}}}\right]^{1-\sigma}\right.$$
$$\left.+\beta\phi\cdot\left\{\frac{\tau+(1+r)\left[\tau-(1+r)b_0\right]-(1+r)\alpha(\tau-rb_0)}{1+\theta^{\frac{1}{\sigma}}}\right\}^{1-\sigma}\right\}, \tag{30}$$
$$V_{O,1}^{r}(b_0,\alpha)=\frac{1}{1-\sigma}\cdot\left\{\left(\theta+\theta^{\frac{1-\sigma}{\sigma}}\right)\cdot\left[\frac{\alpha(\tau-rb_0)}{1+\theta^{\frac{1}{\sigma}}}\right]^{1-\sigma}\right.$$
$$\left.+\beta\phi\cdot\left\{\frac{\tau+(1+r)\left[\tau-(1+r)b_0\right]-(1+r)\alpha(\tau-rb_0)}{1+\theta^{\frac{1}{\sigma}}}\right\}^{1-\sigma}\right\}. \tag{31}$$

Equations (30) and (31) demonstrate that the expenditure rule, denoted as α, exerts dual effects on the value functions $V_{I,1}^{r}(b_0,\alpha)$ and $V_{O,1}^{r}(b_0,\alpha)$, respectively. The initial terms in equations (30) and (31) each function as an increasing factor with respect to α; a more permissive expenditure rule, indicated by a larger α, results in an augmented provision of public goods $\{g_{I,1}^{r*}(b_0,\alpha), g_{O,1}^{r*}(b_0,\alpha)\}$ during period 1. Conversely, the subsequent terms in equations (30) and (31) each behave as a decreasing function of α. A looser expenditure rule corresponds to an increased propensity of the period-1 incumbent party to provide public goods and issue public debt excessively. In turn, this diminishes public goods in period 2. Therefore, an increase in α, accompanied by a relaxation of the expenditure rule, yields both positive and negative impacts on the value associated with both the incumbent and opposition parties. The optimal expenditure rule, which is investigated in the next section, represents the point at which these opposing effects are balanced.

In conclusion, we determine the expenditure rule that maximizes the value functions for both the incumbent and opposition parties. By examining equations (30) and (31), we can derive the conditions that determine the optimal rule for the incumbent and opposition parties, respectively, as follows:

$$\frac{\partial V^r_{I,1}(b_0,\alpha)}{\partial \alpha} \geq 0 \;\Leftrightarrow\; \alpha \leq \frac{\tau + (1+r)\left[\tau - (1+r)b_0\right]}{(\tau - rb_0)\left[(1+r) + \left(\phi/(1+\theta^{\frac{1}{\sigma}})\right)^{\frac{1}{\sigma}}\right]} \equiv \bar{\alpha}(b_0), \tag{32}$$

$$\frac{\partial V^r_{O,1}(b_0,\alpha)}{\partial \alpha} \gtreqless 0 \;\Leftrightarrow\; \alpha \lesseqgtr \frac{\tau + (1+r)\left[\tau - (1+r)b_0\right]}{(\tau - rb_0)\left[(1+r) + \left(\phi/\left(\theta + \theta^{\frac{1-\sigma}{\sigma}}\right)\right)^{\frac{1}{\sigma}}\right]}. \tag{33}$$

Equation (32) demonstrates that, within the range of α where Assumption 3 holds, $V^r_{I,1}(b_0,\alpha)$ is strictly increasing in α. This implies that the incumbent party prefers the loosest possible expenditure rule, i.e., $\alpha = \bar{\alpha}(b_0)$. By contrast, because $\theta < 1$, the right-hand side of equation (33) is smaller than $\bar{\alpha}(b_0)$. As long as Assumption 3 holds, $V^r_{O,1}(b_0,\alpha)$ is a hump-shaped function in α, which implies that the expenditure rule that maximizes the opposition party's value is an interior solution.

4 Optimal Expenditure Rule and Supermajority Threshold

In this section, we derive the optimal combination of the expenditure rule and the supermajority threshold to maximize social welfare by aggregating voters' utilities and analyzing their characteristics. A constitutional designer who represents the voters is assumed to set the levels of the expenditure rule and the supermajority threshold before the number of seats held by the two political parties, p and $1-p$, is determined through elections. This assumption reflects the fact that by the time an election takes place, the fiscal rule and the supermajority threshold are typically already enshrined in the constitution or law in many countries and are rarely amended.

4.1 Social Welfare Function

In constructing the social welfare function, it is important to note that the present model features two types of voters who differ in their preferences for public goods, and each voter lives for two periods. Because the preferences of each voter type are determined probabilistically at each point in time, voters face political risk. To incorporate this risk, the social welfare function is defined as the aggregate expected utility of voters in period 1. Given that parties A and B represent voter types A and B, respectively, the social welfare function is obtained by aggregating the expected utilities of the two parties in period 1.

Without loss of generality, let p denote the number of seats for party A in period 1. As illustrated in Figure 2, party A becomes the incumbent party when $p \geq 1/2$, while party B takes on this role when $p < 1/2$. If $p \geq \delta$ or $p \leq 1-\delta$, the state is categorized as the dictator state; otherwise, it is classified as the rule state.

The variable p serves a dual purpose, representing both the number of seats for party A and number of voters of type A. The magnitude of p directly (inversely) correlates with the number of voters for type A (B). Assuming a Benthamite social welfare function, each political party's value should be weighted by the number of seats. Consequently, we introduce the social welfare function $W(b_0; \delta, \alpha)$, defined as follows.

$$
\begin{aligned}
W(b_0, \delta, \alpha) =& \mathbb{E}\left[pV_{A,1} + (1-p)V_{B,1}\right] \\
=& \underbrace{\int_{\delta}^{1} \cdot \left[pV^d_{I,1}(b_0) + (1-p)V^d_{O,1}(b_0)\right] \mathrm{d}p}_{\text{dictator state under party } A} + \underbrace{\int_{1/2}^{\delta} \cdot \left[pV^r_{I,1}(b_0,\alpha) + (1-p)V^r_{O,1}(b_0,\alpha)\right] \mathrm{d}p}_{\text{rule state under party } A} \\
&+ \underbrace{\int_{1-\delta}^{1/2} \cdot \left[pV^r_{O,1}(b_0;\alpha) + (1-p)V^r_{I,1}(b_0;\alpha)\right] \mathrm{d}p}_{\text{rule state under party } B} + \underbrace{\int_{0}^{1-\delta} \cdot \left[pV^d_{O,1}(b_0) + (1-p)V^d_{I,1}(b_0)\right] \mathrm{d}p}_{\text{dictator state under party } B} \\
=& (1-\delta^2)V^d_{I,1}(b_0) + (1-\delta)^2 V^d_{O,1}(b_0) + \frac{4\delta^2-1}{4} \cdot V^r_{I,1}(b_0,\alpha) - \frac{4(1-\delta)^2-1}{4} \cdot V^r_{O,1}(b_0,\alpha),
\end{aligned}
\tag{34}
$$

where p is uniformly distributed in the $[0, 1]$ interval.

Unlike Piguillem and Riboni (2021), who assume a stationary distribution of political preferences with equal weights, our social welfare function recognizes that seat shares (p) serve as a proxy for the "social mood." A constitutional designer who assigns equal weights regardless of p would be normatively indifferent to the electoral outcome. Instead, we assume the designer seeks to maximize a social welfare function that reflects the democratic weight of the parties, consistent with the Benthamite aggregation in (34). In a stationary world, equal weights may be a reasonable benchmark; however, when society's preferences for public expenditure evolve over time, the constitutional designer should place greater weight on the party that commands broader popular support. By scaling welfare weights with p for party A and $1-p$ for party B, our model captures the idea that a government enjoying a strong electoral mandate should be granted greater fiscal discretion to respond to society's changing demands for public expenditure.[3]

The social welfare function in (34) emphasizes the trade-off faced by the constitutional designer when setting the expenditure rule and the supermajority threshold for approving its suspension. On the one hand, the constitutional designer aims to curb excessive fiscal spending by imposing an expenditure rule on the incumbent party. On the other hand, the designer is incentivized to relax this expenditure rule, as it constrains the incumbent party, which holds significant weight in the social welfare function. If the constitutional designer could set the expenditure rule after observing each party's seat shares, p and $1-p$, she would likely aim to ease the spending constraint on the incumbent party as its seat share, p, increases.

In the current framework, however, the constitutional designer must establish an expenditure rule that maximizes social welfare before knowing the seat distribution between the two parties, p and $1-p$. This assumption enables us to emphasize the role of the supermajority threshold in approving the expenditure rule's suspension. Specifically, the constitutional designer allows suspending the expenditure rule only if the incumbent party's seat share exceeds a threshold, δ. In other words, instead of adjusting the expenditure rule according to the incumbent's seat share, p, the rule can only be suspended when p exceeds the supermajority threshold. In this way, the supermajority threshold δ balances two objectives: imposing fiscal discipline on governments with moderate popular support to curb excessive spending, while allowing governments with strong mandates to respond flexibly to society's changing demands for public expenditure. The following analysis shows that this discontinuous suspension mechanism, reflecting legislative decision-making processes in many developed countries, serves as a key factor in maximizing social welfare.

In the following analysis, we initially determine the optimal expenditure rule α corresponding

[3] A potential concern is whether parliamentary seat shares (p and $1-p$) accurately reflect the "social mood" or the true preferences of the electorate. We assume that electoral outcomes are the primary institutional signal of societal shifts. While the agency relationship between voters and representatives is complex, the constitutional designer must rely on observed mandates to calibrate the trade-off between fiscal discipline and democratic responsiveness. In this sense, p represents the most legible metric of political legitimacy in a representative democracy.

to a given supermajority threshold δ (in Section 4.2). Then, we identify the optimal δ for a given α (in Section 4.3). Finally, we investigate the optimal combination of α and δ (in Section 4.4).

4.2 Optimal Expenditure Rule

Consider the optimal expenditure rule $\alpha^*(b_0, \delta)$ with a given supermajority threshold δ. If $\delta = 1/2$, the incumbent party acts like a dictator regardless of seat count, rendering the expenditure rule entirely meaningless. Consequently, we concentrate on cases where $\delta > 1/2$ and derive the following lemma.

Lemma 2 *Suppose a supermajority threshold $\delta \in (1/2, 1]$ is provided. The optimal expenditure rule $\alpha^*(b_0, \delta)$ is then determined by:*

$$\alpha^*(b_0, \delta) = \frac{\tau + (1+r) \cdot [\tau - (1+r)b_0]}{\tau - rb_0} \cdot \frac{\Gamma(\delta)}{1 + (1+r)\Gamma(\delta)} \in (0, \bar{\alpha}(b_0)) \quad \forall \delta \in (1/2, 1], \tag{35}$$

where $\Gamma(\delta)$ is defined by

$$\Gamma(\delta) \equiv \left[1 + \frac{(2\delta - 1)\left(1 + \theta^{\frac{1}{\sigma}} - \theta - \theta^{\frac{1-\sigma}{\sigma}}\right)}{2\left(1 + \theta^{\frac{1}{\sigma}} + \theta + \theta^{\frac{1-\sigma}{\sigma}}\right)} \right]^{\frac{1}{\sigma}} > 1. \tag{36}$$

Proof. See Appendix B.

Lemma 2 indicates that for any supermajority threshold $\delta \in (1/2, 1]$, the optimal interior solution for the expenditure rule is determined as follows. As observed in equation (34), if a dictator state exists, the expenditure rule does not affect social welfare. The optimal expenditure rule is defined to maximize expected welfare when the rule state occurs. In such cases, the incumbent party prefers the loosest possible expenditure rule ($\alpha = \bar{\alpha}(b_0)$) (see (32) and (33)). Conversely, the opposition party prefers a specific interior solution for the expenditure rule. Analyzing both scenarios shows that the expenditure rule $\alpha^*(b_0, \delta)$ that maximizes social welfare is obtained as an interior solution.

The implication of this result is as follows: Lemma 1's result suggests that the incumbent party has an incentive to overspend with political conflict in the provision of public goods. Stricter expenditure rules can mitigate such excessive expenditure and improve the welfare of the opposition's supporters. However, tighter expenditure rules worsen the incumbent party supporters' welfare. Therefore, the optimal expenditure rule is the one with an interior point that effectively balances these trade-offs.

The result presented in Lemma 2 reveals that the maximum allowable expenditure, denoted as $\alpha^*(b_0, \delta) \cdot (\tau - rb_0)$ under the optimal expenditure rule, is a decreasing function of b_0. Thus, imposing stricter expenditure restrictions is optimal when the initial public debt balance is larger. This finding highlights a link between a country's fiscal position at the time of adopting a fiscal rule and the design of the rule itself.

Moreover, $\alpha^*(b_0, \delta)$ is an increasing function of δ, indicating that the tighter the supermajority rule (demanding greater compliance with the expenditure rule), the more lenient the optimal expenditure rule becomes. In other words, the strictness of expenditure rules and the constraints on overrides of the rules are substitutable. When the supermajority threshold is stringent, even the incumbent party with many seats is bound by an expenditure rule. In such cases, many voters supporting the incumbent party are constrained by expenditure rules. This implies that the welfare loss associated with enforcing a stringent expenditure rule becomes significant, making a more lenient expenditure rule optimal.

4.3 Optimal Supermajority Threshold

Next, given the expenditure rule α, we find the optimal supermajority threshold $\delta^*(b_0, \alpha)$.

Lemma 3 *For a given expenditure rule $\alpha \in [0, \bar{\alpha}(b_0))$, the optimal supermajority threshold $\delta^*(b_0, \alpha)$ is:*

$$\delta^*(b_0, \alpha) = \frac{V^r_{O,1}(b_0, \alpha) - V^d_{O,1}(b_0)}{[V^d_{I,1}(b_0) - V^r_{I,1}(b_0, \alpha)] + [V^r_{O,1}(b_0, \alpha) - V^d_{O,1}(b_0)]} < 1. \tag{37}$$

In addition, $\delta^*(b_0, \alpha) > 1/2$ holds if the following condition is satisfied:

$$V^d_{I,1}(b_0) - V^r_{I,1}(b_0, \alpha) < V^r_{O,1}(b_0, \alpha) - V^d_{O,1}(b_0). \tag{38}$$

Proof. See Appendix C.

Lemma 3 shows that, for any expenditure rule $\alpha < \bar{\alpha}(b_0)$, under specific conditions, the optimal supermajority threshold is determined by the interior point. If δ is small and the supermajority rule is loose, the dictator state is likely, rendering the expenditure rule meaningless. This results in over-expenditure by the incumbent party and welfare losses for the opposition. Strengthening the expenditure rule's enforceability involves implementing more stringent supermajority rules, curbing excessive fiscal expenditure, and improving opposition supporters' welfare. However, this comes at the cost of worsening the incumbent party supporters' welfare. Therefore, the optimal supermajority threshold delicately balances these tradeoffs, offering a nuanced solution that mitigates excessive expenditures while considering the welfare of both parties.

Next, consider the condition in (38) for $\delta^*(b_0, \alpha) > 1/2$ to be satisfied. If $\delta = 1/2$, the incumbent party, holding more than half of the seats, can consistently suspend the expenditure rule. When (38) holds, and therefore, $\delta^*(b_0, \alpha) > 1/2$, the utility gain experienced by the opposition party's supporters during the transition from the dictator to the rule state $(V^r_{O,1}(b_0, \alpha) - V^d_{O,1}(b_0))$ exceeds the utility loss for the incumbent party's supporters $(V^d_{I,1}(b_0) - V^r_{I,1}(b_0, \alpha))$. Given that the incumbent party supporters exceed the opposition party supporters, exempting from the expenditure rule at all times becomes optimal from the social welfare maximization perspective if (38) is violated.

4.4 Optimal Combination of Expenditure Rule and Supermajority Threshold

Based on the results in Sections 4.2 and 4.3, we seek to identify the optimal combination of expenditure rule and supermajority threshold that maximizes social welfare. Formally, we define the optimal combination as follows.

Definition 1 *An optimal combination of expenditure rule and supermajority threshold is represented by a pair $\{\alpha^{opt}(b_0), \delta^{opt}(b_0)\}$ that simultaneously satisfies the following two equations:*

$$\alpha^{opt}(b_0) = \alpha^*(b_0, \delta^{opt}), \quad \delta^{opt}(b_0) = \delta^*(b_0, \alpha^{opt}). \tag{39}$$

From Lemmas 2 and 3, we obtain the following proposition.

Proposition 1 *There is an interior optimal combination that satisfies $\alpha^{opt}(b_0) \in (0, \bar{\alpha}(b_0))$ and $\delta^{opt}(b_0) \in (1/2, 1)$.*

Proof. See Appendix D.

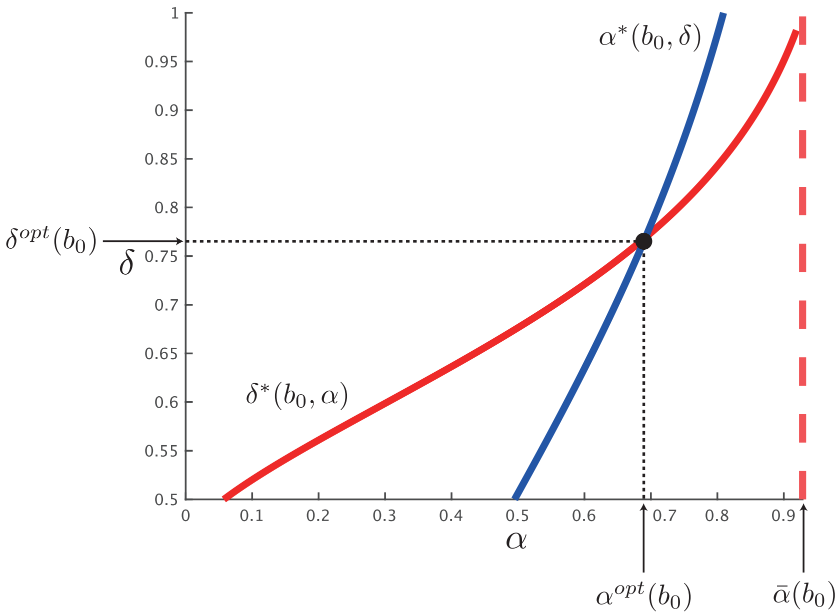

Figure 3: Numerical calculations of the optimal expenditure rule $\alpha^*(b_0, \delta)$ given the supermajority rule δ, and optimal supermajority rule $\delta^*(b_0, \alpha)$ given the expenditure rule α. Note: $b_0 = 1$, $\sigma = 0.1$, $r = 0.02$ ($\beta = 1/1.02$), $\theta = 0.5$, $\tau = 1$.

Figure 3 illustrates the reaction functions of $\delta^*(b_0, \alpha)$ and $\alpha^*(b_0, \delta)$ for a given initial public debt b_0. The intersection of the two reaction functions is the optimal combination of expenditure rule and supermajority threshold $\{\alpha^{opt}(b_0), \delta^{opt}(b_0)\}$ that satisfies Definition 1. In the numerical example depicted in Figure 3, $\alpha^*(b_0, \delta)$ is an increasing function of δ, while $\delta^*(b_0, \alpha)$ is an increasing function of α. This finding highlights that a looser expenditure rule is optimal under a tighter supermajority rule, whereas a tighter supermajority rule is optimal under a looser expenditure rule. Thus, it represents a substitutable relationship between expenditure rule and supermajority threshold.

Figure 3 demonstrates that the optimal combination of expenditure rule and supermajority threshold at the interior point is attained, as indicated by Proposition 1. The optimal supermajority threshold satisfies $1/2 < \delta^{opt}(b_0) < 1$. The requirement of $\delta^{opt}(b_0) < 1$ implies that a rule prohibiting all suspensions ($\delta = 1$) is not optimal. If no suspensions are allowed, the incumbent party, which has many supporters, will be forced to adhere to the expenditure rules. Thus, a rule that causes welfare loss for most members of a society cannot be deemed optimal. The requirement of $1/2 < \delta^{opt}(b_0)$ suggests that using a simple majority rule to determine the suspension of expenditure rules–a common practice in many developed countries–is not desirable.

Instead, these countries should adopt a more stringent supermajority threshold for suspending expenditure rules.

5 Comparative Statics Analysis

We have analyzed the optimal combination of expenditure rule and supermajority threshold, considering the initial public debt balance b_0 and the degree of political conflict to public expenditure among voters denoted as θ as predetermined. These parameters signify variations in each country's fiscal environment substantially influence the country's optimal expenditure rule and supermajority threshold. In this section, we investigate the impact of the initial public debt balance b_0 and political conflict θ on the formulation of the optimal combination of the expenditure rule and supermajority threshold through comparative statics analysis.

5.1 Effects of Initial Public Debt Balance

We examine the impact of the initial public debt balance b_0 on the optimal combination of expenditure rule and supermajority threshold, $\alpha^{opt}(b_0)$ and $\delta^{opt}(b_0)$. The results of the comparative statics analysis are summarized through the following proposition.

Proposition 2 *The optimal supermajority threshold δ^{opt} remains independent of b_0, whereas the optimal expenditure rule $\alpha^{opt}(b_0)$ is decreasing in b_0.*

Proof. See Appendix E.

Figure 4 illustrates the relationship between b_0 and $\{\alpha^{opt}(b_0), \delta^{opt}\}$ as presented in Proposition 2, supported by numerical examples. The intuition underlying the independence of δ^{opt} from b_0 is as follows. The supermajority threshold governs the balance between fiscal flexibility and the cost of politically biased expenditure. A change in the initial debt level alters the total fiscal resources available, but under CRRA preferences it affects both sides of this trade-off proportionally, leaving the optimal balance point unchanged.

Formally, define $T(b_0) \equiv \tau + (1+r)[\tau - (1+r)b_0]$, which represents the government's net wealth, i.e., the present value of total tax revenues over the two periods net of the initial debt

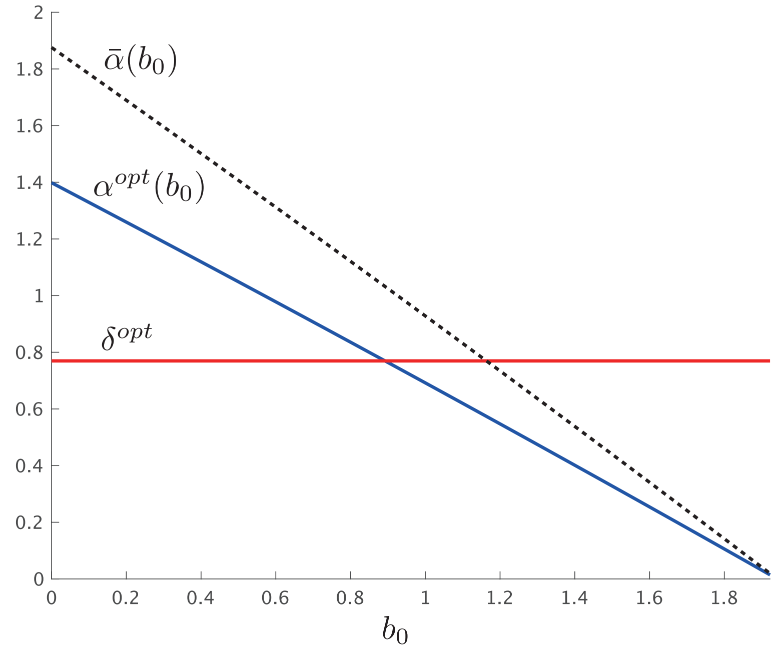

Figure 4: Illustration of $\alpha^{opt}(b_0)$ and δ^{opt} with b_0 on the horizontal axis. For other parameters, $\sigma = 0.1$, $r = 0.02$ ($\beta = 1/1.02$), $\theta = 0.5$ and $\tau = 1$.

obligations. With CRRA preferences, the value functions evaluated at α^* take the form

$$V^d_{I,1}(b_0) = C^d_I \cdot (T(b_0))^{1-\sigma}\,, \tag{40}$$

$$V^r_{I,1}(b_0, \alpha^*(b_0, \delta)) = C^r_I(\delta) \cdot (T(b_0))^{1-\sigma}\,, \tag{41}$$

where, C^d_I and $C^r_I(\delta)$ are, using equations (21) and (30), defined as:

$$C^d_I = \frac{1 + \theta^{\frac{1}{\sigma}} + \beta(1 + \theta^{\frac{1}{\sigma}}) \left(\frac{\phi}{1+\theta^{\frac{1}{\sigma}}}\right)^{\frac{1}{\sigma}}}{(1-\sigma)\left[\left((1+r) + \left(\frac{\phi}{1+\theta^{\frac{1}{\sigma}}}\right)^{\frac{1}{\sigma}}\right)\left(1 + \theta^{\frac{1}{\sigma}}\right)\right]^{1-\sigma}}, \tag{42}$$

$$C^r_I(\delta) = \frac{(1 + \theta^{\frac{1}{\sigma}}) \cdot \Gamma(\delta)^{1-\sigma} + \beta\phi}{(1-\sigma)\left[(1 + (1+r)\Gamma(\delta))\left(1 + \theta^{\frac{1}{\sigma}}\right)\right]^{1-\sigma}}. \tag{43}$$

Analogous expressions hold for $V^d_{O,1}$ and $V^r_{O,1}$. That is, all value functions are proportional to $(T(b_0))^{1-\sigma}$, with coefficients that do not depend on b_0. Substituting into equation (37), the common factor $(T(b_0))^{1-\sigma}$ cancels from both the numerator and denominator, so the fixed point δ^{opt} is independent of b_0.[4]

[4]This property holds more generally whenever the utility function $u(g_A, g_B)$ is homogeneous of degree k: in such cases, all value functions are proportional to $(T(b_0))^k$, and this common factor cancels out in the ratio that

Regarding the expenditure rule, $\alpha^{opt}(b_0)$ shows a decreasing trend for b_0, indicating that a more restrictive expenditure rule is optimal for higher initial public debt levels. This trend arises because as b_0 increases, stronger constraints on excessive public expenditure are necessary to meet the terminal condition $b_2 = 0$. Therefore, a higher initial public debt balance requires an optimal expenditure rule with tighter restrictions.[5]

By Proposition 2, the optimal expenditure rule $\alpha^{opt}(b_0)$ is contingent upon the initial public debt balance (b_0), whereas the optimal supermajority threshold δ^{opt} establishes its optimal level independently of the initial public debt balance. This implies that policymakers should adjust the optimal expenditure rule in response to current fiscal conditions. Once the optimal level of the supermajority threshold is determined, it ought to be maintained irrespective of the fiscal situation. This result implies that it is socially beneficial to establish a mechanism where the expenditure rule is flexibly adjusted by law, while the supermajority threshold is constitutionally administered over the long term. This implication derived from the present analysis represents one of the significant conclusions, addressing a dimension not explored in previous studies.

5.2 Effects of Differences in Preferences

The parameter θ represents the disparity in preferences between the incumbent and opposition parties. A smaller value of θ indicates a more substantial difference in preferences, highlighting increased conflict between the incumbent and opposition parties. Since θ functions as a parameter characterizing the shape of the voters' (parties') utility function, its modification not only affects the equilibrium supply of public goods and issuance of public debt but also shapes social welfare through utility. This impact complicates the derivation of the closed-form solution. To address this complexity, we turn to a numerical example to analyze how a change in θ influences the optimal expenditure rule and the optimal supermajority threshold.

Figure 5 illustrates a numerical example depicting the relationship between θ and $\{\alpha^{opt}(b_0), \delta^{opt}\}$. As θ decreases, $\alpha^{opt}(b_0)$ increases and δ^{opt} decreases. That is, stronger conflict between parties corresponds to more relaxed optimal expenditure and supermajority rules. The underlying mechanism is as follows: When θ decreases and party conflict intensifies, the change in incumbent

determines δ^* in equation (37), leaving δ^* unchanged. The CRRA specification adopted in this paper, where $k = 1 - \sigma$, is a leading example of this class.

[5] This implication might not change even if an infinite-period model is considered; even in an infinite-period model, the terminal condition that the outstanding public debt must not exceed the natural debt limit must be satisfied.

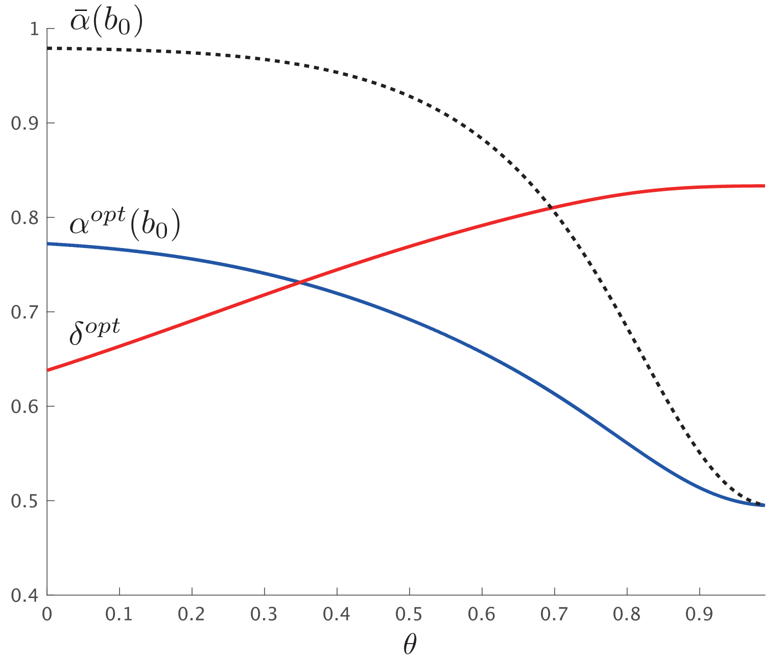

Figure 5: Illustration of $\alpha^{opt}(b_0)$ and δ^{opt} with θ on the horizontal axis. The horizontal axis ranges from $\theta = 0$ to $\theta = 0.99$, consistent with the assumption $\theta < 1$. For other parameters, $\sigma = 0.1$, $r = 0.02$ ($\beta = 1/1.02$), $b_0 = 1$, and $\tau = 1$.

party supporters' preferences exerts a stronger influence on social welfare than that in opposition party supporters' preferences. As θ decreases, the optimal fiscal rule that maximizes social welfare becomes more oriented towards the incumbent party. Drawing on the insights from Lemma 1, the heightened conflict between political parties increases public expenditure in the dictator state, where the expenditure rule is irrelevant. Consequently, a smaller θ leads to a more relaxed optimal expenditure rule. Moreover, an optimal looser supermajority rule is preferred, because a stricter supermajority rule adversely affects the incumbent party supporters' welfare.[6]

6 Conclusion

This study theoretically examines the optimal supermajority threshold's characteristics based on the expenditure rule, assuming that the latter imposes a public expenditure ceiling. We focus on scenarios where suspensions of the expenditure rule are allowed if the incumbent party holds more than a threshold share of seats in parliament, and we investigate the optimal combination

[6]In the limiting case $\theta = 1$, the incumbent and opposition parties value public goods identically, so the expenditure rule ceases to affect social welfare and any $\delta \in [1/2, 1]$ is optimal. Consequently, the uniqueness of δ^{opt} is lost at $\theta = 1$, and the reported values of δ^{opt} at $\theta = 1$ in Figure 5 should be interpreted with caution near this boundary.

of the expenditure rules and the supermajority threshold from the perspective of social welfare maximization. The primary findings are as follows: First, an expenditure rule prohibiting any suspensions is socially undesirable. When skewed voter selection favors the incumbent party with a significant seat majority, temporarily relaxing the expenditure rule may prove beneficial. In addition, using a simple majority rule to determine the suspension of expenditure rules–a common practice in many developed countries–is not desirable. Instead, a more stringent supermajority threshold for suspending expenditure rules is required from the standpoint of social welfare maximization.

Second, the optimal supermajority threshold remains independent of outstanding public debt. This is because public debt affects social welfare only through public expenditure levels. While the supermajority threshold dictates adherence to the expenditure rule, public expenditure is regulated by the expenditure rule. Thus, public debt influences the optimal expenditure rule but not the optimal supermajority threshold. This underscores the necessity of adjusting the expenditure rule in line with current fiscal conditions while the optimal supermajority threshold remains constant regardless of the fiscal circumstances once established.

Third, greater disparities in preferences for public goods among voters (or between parties) favor a looser supermajority threshold. As the incumbent party's supporters outnumber those of the opposition, political conflicts tilt the optimal supermajority threshold toward the incumbent party's favor, warranting a looser deviance rule.

Several issues warrant further investigation. First, while we examine the optimal supermajority threshold for the expenditure rule, similar analyses can explore other fiscal rule types. For example, studying the optimal supermajority threshold characteristics for rules governing the public debt ceiling in EU countries can be insightful. Second, while we base suspensions on the incumbent party's number of seats, in practice, even with a significant seat majority, a suspension often requires agreement from the opposition party. Addressing this can be beneficial by incorporating negotiations between the ruling and opposition parties into the model. Lastly, although we employ a two-period model for analytical tractability, extending it to an infinite-horizon version can facilitate analyzing the relationship between the supermajority threshold and public debt balance dynamics.

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Appendix

A Proof of Lemma 1

It is sufficient to show that $\left(g_{I,1}^{d*}(b_0)+g_{O,1}^{d*}(b_0)\right)$ is a decreasing function of θ. From equations (18) and (19), the government expenditure in the dictator state is:

$$g_{I,1}^{d*}(b_0)+g_{0,1}^{d*}(b_0)=\frac{\tau+(1+r)\left[\tau-(1+r)b_0\right]}{(1+r)+\left(\phi/(1+\theta^{\frac{1}{\sigma}})\right)^{\frac{1}{\sigma}}}. \tag{A.1}$$

In the expression of (A.1), θ is included in the term $\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)$. Given the definition of ϕ in (16), the differentiation of $\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)$ with respect to θ leads to:

$$\frac{\partial}{\partial\theta}\left(\frac{\phi}{1+\theta^{\frac{1}{\sigma}}}\right)=\frac{1-\frac{1-\sigma}{\sigma}\cdot\theta^{\frac{1}{\sigma}}+\frac{1-\sigma}{\sigma}\cdot\theta^{\frac{1}{\sigma}-2}-\theta^{\frac{2}{\sigma}-2}}{2\left(1+\theta^{\frac{1}{\sigma}}\right)^2}=\frac{1-\theta^{\frac{2(1-\sigma)}{\sigma}}+\frac{1-\sigma}{\sigma}\cdot\theta^{\frac{1}{\sigma}}\left(\frac{1}{\theta^2}-1\right)}{2\left(1+\theta^{\frac{1}{\sigma}}\right)^2}>0,$$

showing that $\left(g_{I,1}^{d*}(b_0)+g_{O,1}^{d*}(b_0)\right)$ in (A.1) is decreasing in θ.

Q.E.D.

B Proof of Lemma 2

Suppose that $\delta>1/2$ holds. The first partial derivative of $W(b_0,\delta,\alpha)$ in equation (34) with respect to α is:

$$\begin{aligned}\frac{\partial W(b_0,\delta,\alpha)}{\partial\alpha}=&\left[\frac{4\delta^2-1}{4}\cdot\left(1+\theta^{\frac{1}{\sigma}}\right)+\frac{-4\delta^2+8\delta-3}{4}\cdot\left(\theta+\theta^{\frac{1-\sigma}{\sigma}}\right)\right]\cdot\alpha^{-\sigma}\cdot\left(\frac{\tau-rb_0}{1+\theta^{\frac{1}{\sigma}}}\right)^{1-\sigma}\\&+\left(\frac{4\delta^2-1}{4}+\frac{-4\delta^2+8\delta-3}{4}\right)\cdot\beta\phi\cdot\left\{\frac{\tau+(1+r)\left[\tau-(1+r)b_0\right]-(1+r)\alpha(\tau-rb_0)}{1+\theta^{\frac{1}{\sigma}}}\right\}^{-\sigma},\end{aligned}$$

or,

$$\begin{aligned}\frac{\partial W(b_0,\delta,\alpha)}{\partial\alpha}=&\frac{(2\delta-1)(\tau-rb_0)}{1+\theta^{\frac{1}{\sigma}}}\cdot\left\{\frac{(2\delta-1)\left(1+\theta^{\frac{1}{\sigma}}-\theta-\theta^{\frac{1-\sigma}{\sigma}}\right)+4\phi}{4}\cdot\left[\frac{\alpha(\tau-rb_0)}{1+\theta^{\frac{1}{\sigma}}}\right]^{-\sigma}\right.\\&\left.-\phi\cdot\left\{\frac{\tau+(1+r)\left[\tau-(1+r)b_0\right]-(1+r)\alpha(\tau-rb_0)}{1+\theta^{\frac{1}{\sigma}}}\right\}^{-\sigma}\right\},\end{aligned} \tag{B.1}$$

where we use $\beta(1+r)=1$ in Assumption 2 to derive the expression in (B.1). The second partial derivative of $W(b_0,\delta,\alpha)$ with respect to α is:

$$\frac{\partial^2W(b_0,\delta,\alpha)}{\partial\alpha^2}=\frac{(2\delta-1)(\tau-rb_0)}{1+\theta^{\frac{1}{\sigma}}}\cdot(-\sigma)\cdot\left\{\frac{(2\delta-1)\left(1+\theta^{\frac{1}{\sigma}}-\theta-\theta^{\frac{1-\sigma}{\sigma}}\right)+4\phi}{4}\cdot\alpha^{-\sigma-1}\cdot\left(\frac{\tau-rb_0}{1+\theta^{\frac{1}{\sigma}}}\right)^{-\sigma}\right.$$

$$+\phi \cdot \left\{ \frac{\tau + (1+r)\left[\tau - (1+r)b_0\right] - (1+r)\alpha(\tau - rb_0)}{1+\theta^{\frac{1}{\sigma}}} \right\}^{-\sigma-1} \cdot \frac{(1+r)(\tau - rb_0)}{1+\theta^{\frac{1}{\sigma}}} \Bigg\}.$$

From (12), $\left(1+\theta^{\frac{1}{\sigma}} - \theta - \theta^{\frac{1-\sigma}{\sigma}}\right) > 0$ is satisfied; and according to Assumption 1, $(\tau - rb_0 > 0)$ is also satisfied. Furthermore, Assumption 1 implies $b_0 < b^{NDL}$, and Assumption 3 ensures $\alpha < \bar{\alpha}(b_0)$. Therefore, the following expression holds:

$$\begin{aligned}
&\tau + (1+r)\left[\tau - (1+r)b_0\right] - (1+r)(\tau - rb_0)\alpha \\
&\quad > \tau + (1+r)\left[\tau - (1+r)b_0\right] - (1+r)(\tau - rb_0)\bar{\alpha}(b_0); \text{ since } \alpha < \bar{\alpha}(b_0) \\
&\quad = \left\{\tau + (1+r)\left[\tau - (1+r)b_0\right]\right\} \cdot \frac{\left(\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}}}{(1+r) + \left(\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}}}; \text{ from the definition of } \bar{\alpha}(b_0) \\
&\quad > \left\{\tau + (1+r)\left[\tau - (1+r)b^{NDL}\right]\right\} \cdot \frac{\left(\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}}}{(1+r) + \left(\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}}}; \text{ since } b_0 < b^{NDL} \\
&\quad = 0. \qquad \text{(B.2)}
\end{aligned}$$

Given that we are currently examining the case where $\delta > 1/2$, $(\partial^2 W(b_0;\delta,\alpha))/(\partial\alpha^2) < 0$ is affirmed based on the aforementioned formula. Consequently, the second-order condition for optimality is satisfied.

Next, we consider the first-order condition for optimality. The optimal level of the expenditure rule for a given δ, denoted as α^*, satisfies $\partial W(b_0,\delta,\alpha^*)/\partial\alpha = 0$, or:

$$\Gamma(\delta) \cdot \frac{\tau + (1+r)\left[\tau - (1+r)b_0\right] - (1+r)\alpha^*(\tau - rb_0)}{1+\theta^{\frac{1}{\sigma}}} = \alpha^* \cdot \left[\frac{\tau - rb_0}{1+\theta^{\frac{1}{\sigma}}}\right],$$

where $\Gamma(\delta)$ is defined as follows:

$$\Gamma(\delta) \equiv \left[\frac{(2\delta - 1)\left(1+\theta^{\frac{1}{\sigma}} - \theta - \theta^{\frac{1-\sigma}{\sigma}}\right) + 4\phi}{4\phi}\right]^{\frac{1}{\sigma}} \geq 1. \qquad \text{(B.3)}$$

$\Gamma(\delta) \geq 1$ holds because (i) we assume $\delta > 1/2$ and (ii) $1+\theta^{\frac{1}{\sigma}} - \theta - \theta^{\frac{1-\sigma}{\sigma}} > 0$ holds from (12). The above first-order condition for optimality is solved for α^* as follows:

$$\alpha^*(b_0,\delta) = \frac{\tau + (1+r)\cdot\left[\tau - (1+r)b_0\right]}{\tau - rb_0} \cdot \frac{\Gamma(\delta)}{1+(1+r)\Gamma(\delta)}.$$

In the expression in (B.3), the following two properties hold under Assumption 1:

$$\tau + (1+r)\cdot\left[\tau - (1+r)b_0\right] > \tau + (1+r)\cdot\left[\tau - (1+r)b^{NDL}\right] = 0, \qquad \text{(B.4)}$$

$$\tau - rb_0 > \tau - rb^{NDL} = \frac{1}{(1+r)^2} > 0, \qquad \text{(B.5)}$$

These two properties ensure that $\alpha^*(b_0,\delta) > 0$. Additionally, with Assumption 3 and the conditions in (B.4) and (B.5), the following properties are fulfilled:

$$\alpha^*(b_0,\delta) < \bar{\alpha}(b_0) \;\Leftrightarrow\; \Gamma(\delta)\cdot\left(\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}} < 1 \Leftrightarrow\; \delta < \frac{3}{2}.$$

Thus, for any $\delta \in (1/2, 1]$, $\alpha^*(b_0,\delta) < \bar{\alpha}(b_0)$ holds.

Q.E.D.

C Proof of Lemma 3

Suppose that $\alpha < \bar{\alpha}(b_0)$ holds. The first partial derivative of $W(b_0,\delta^*,\alpha)$ in (34) with respect to δ is:

$$\frac{\partial W(b_0,\delta^*,\alpha)}{\partial \delta} = -2\cdot\left[\delta^* V^d_{I,1}(b_0) + (1-\delta^*)V^d_{O,1}(b_0) - \delta^* V^r_{I,1}(b_0,\alpha) - (1-\delta^*)V^r_{O,1}(b_0,\alpha)\right]. \tag{C.1}$$

The second partial derivative of $W(b_0,\delta,\alpha)$ with respect to δ is:

$$\frac{\partial^2 W(b_0,\delta,\alpha)}{\partial \delta^2} = -2\cdot\left[V^d_{I,1}(b_0) - V^r_{I,1}(b_0,\alpha) + V^r_{O,1}(b_0,\alpha) - V^d_{O,1}(b_0)\right]. \tag{C.2}$$

Under Assumption 3, $\alpha < \bar{\alpha}(b_0)$ holds; and $1+\theta^{\frac{1}{\sigma}}-\theta-\theta^{\frac{1-\sigma}{\sigma}} > 0$ holds from (12). Given these two properties with equations (21), (22),(30), and (31), we obtain:

$$\begin{aligned}
&V^d_{I,1}(b_0) - V^r_{I,1}(b_0,\alpha) + V^r_{O,1}(b_0,\alpha) - V^d_{O,1}(b_0)\\
&\quad= \frac{1}{1-\sigma}\cdot\left(1+\theta^{\frac{1}{\sigma}}-\theta-\theta^{\frac{1-\sigma}{\sigma}}\right)\cdot\left[\frac{\bar{\alpha}(b_0)(\tau-rb_0)}{1+\theta^{\frac{1}{\sigma}}}\right]^{1-\sigma}\\
&\qquad - \frac{1}{1-\sigma}\cdot\left(1+\theta^{\frac{1}{\sigma}}-\theta-\theta^{\frac{1-\sigma}{\sigma}}\right)\cdot\left[\frac{\alpha(\tau-rb_0)}{1+\theta^{\frac{1}{\sigma}}}\right]^{1-\sigma}\\
&\quad= \frac{\left(1+\theta^{\frac{1}{\sigma}}-\theta-\theta^{\frac{1-\sigma}{\sigma}}\right)(\tau-rb_0)^{1-\sigma}}{(1-\sigma)\left(1+\theta^{\frac{1}{\sigma}}\right)^{1-\sigma}}\cdot\left[\bar{\alpha}(b_0)^{1-\sigma}-\alpha^{1-\sigma}\right]\\
&\quad> 0.
\end{aligned} \tag{C.3}$$

Expressions in (C.2) and (C.3) ensure that the second derivative in (C.2) is negative, implying that the second-order condition for optimality holds. Thus, from equation (C.1), the optimal δ is:

$$\delta^*(b_0,\alpha) = \frac{V^r_{O,1}(b_0,\alpha) - V^d_{O,1}(b_0)}{[V^d_{I,1}(b_0) - V^r_{I,1}(b_0,\alpha)] + [V^r_{O,1}(b_0,\alpha) - V^d_{O,1}(b_0)]}.$$

Next, we examine the condition for $\delta^*(b_0,\alpha) < 1$ to be satisfied. Considering the property in (C.3), we obtain the following:

$$\delta^*(b_0, \alpha) < 1 \Leftrightarrow V^d_{I,1}(b_0) - V^r_{I,1}(b_0, \alpha) > 0.$$

From the expression in (32), $V^r_{I,1}(b_0; \alpha)$ is monotonically increasing in α under Assumption 3. Using the definition of $\bar{\alpha}(b_0)$ in Assumption 3, we obtain $V^r_{I,1}(b_0, \bar{\alpha}(b_0)) = V^d_{I,1}(b_0)$. Hence, for any $\alpha < \bar{\alpha}(b_0)$, it holds that $0 < V^d_{I,1}(b_0) - V^r_{I,1}(b_0, \alpha)$. Put differently, $\delta^*(b_0, \alpha) < 1$ holds as long as Assumption 3 is satisfied.

Finally, expression in (C.3) leads to the following condition.

$$\delta^*(b_0, \alpha) > \frac{1}{2} \Leftrightarrow V^d_{I,1}(b_0) - V^r_{I,1}(b_0, \alpha) < V^r_{O,1}(b_0, \alpha) - V^d_{O,1}(b_0).$$

Q.E.D.

D Proof of Proposition 1

From equations (35) and (37), the optimal combination of expenditure rule and supermajority threshold, denoted as α^{opt} and δ^{opt}, is determined by solving the following simultaneous equations.

$$\delta = \frac{V^r_{O,1}(b_0, \alpha) - V^d_{O,1}(b_0)}{[V^d_{I,1}(b_0) - V^r_{I,1}(b_0, \alpha)] + [V^r_{O,1}(b_0, \alpha) - V^d_{O,1}(b_0)]}, \tag{D.1}$$

$$\alpha = \bar{\alpha}(b_0) \cdot \left[(1+r) + \left(\phi / \left(1 + \theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}}\right] \cdot \frac{\Gamma(\delta)}{1 + (1+r)\Gamma(\delta)}. \tag{D.2}$$

To find α and δ that satisfy (D.1) and (D.2), we first reformulate both the numerator and denominator of equation (D.1). By considering (22), (31), and (D.2), we can rewrite the numerator of (D.1) as follows:

$$\begin{aligned}
&V^r_{O,1}(b_0, \alpha) - V^d_{O,1}(b_0) \\
&= \frac{1}{1-\sigma} \cdot \left\{ \left(\theta + \theta^{\frac{1-\sigma}{\sigma}}\right) \cdot \left[\frac{\alpha(\tau - rb_0)}{1 + \theta^{\frac{1}{\sigma}}}\right]^{1-\sigma} \right. \\
&\quad \left. + \beta\phi \cdot \left\{ \frac{\tau + (1+r)\left[\tau - (1+r)b_0\right] - (1+r)\alpha(\tau - rb_0)}{1 + \theta^{\frac{1}{\sigma}}} \right\}^{1-\sigma} \right\} \\
&\quad - \frac{1}{1-\sigma} \cdot \left[\theta + \theta^{\frac{1-\sigma}{\sigma}} + \beta(1 + \theta^{\frac{1}{\sigma}}) \left(\frac{\phi}{1 + \theta^{\frac{1}{\sigma}}}\right)^{\frac{1}{\sigma}}\right] \cdot \left[\frac{\bar{\alpha}(b_0)(\tau - rb_0)}{1 + \theta^{\frac{1}{\sigma}}}\right]^{1-\sigma},
\end{aligned}$$

or,

$$\begin{aligned}
&V^r_{O,1}(b_0, \alpha) - V^d_{O,1}(b_0) \\
&= \frac{1}{1-\sigma} \cdot \left[\frac{\bar{\alpha}(b_0)(\tau - rb_0)}{(1 + \theta^{\frac{1}{\sigma}})\left[1 + (1+r)\Gamma(\delta)\right]}\right]^{1-\sigma} \cdot \left\{ \left(\theta + \theta^{\frac{1-\sigma}{\sigma}}\right) \cdot \left[(1+r) + \left(\phi / \left(1 + \theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}}\right]^{1-\sigma} \cdot \Gamma(\delta)^{1-\sigma} \right.
\end{aligned}$$

$$+\beta\phi\cdot\left[(1+r)+\left(\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}}\right]^{1-\sigma}-[1+(1+r)\Gamma(\delta)]^{1-\sigma}\cdot\left[\theta+\theta^{\frac{1-\sigma}{\sigma}}+\beta(1+\theta^{\frac{1}{\sigma}})\left(\frac{\phi}{1+\theta^{\frac{1}{\sigma}}}\right)^{\frac{1}{\sigma}}\right]\Bigg\}. \tag{D.3}$$

Next, using (C.3) and (D.2), we can rewrite the denominator of (D.1) as follows:

$$[V^d_{I,1}(b_0)-V^r_{I,1}(b_0,\alpha)]+[V^r_{O,1}(b_0,\alpha)-V^d_{O,1}(b_0)]=\frac{\left(1+\theta^{\frac{1}{\sigma}}-\theta-\theta^{\frac{1-\sigma}{\sigma}}\right)(\tau-rb_0)^{1-\sigma}}{(1-\sigma)\left(1+\theta^{\frac{1}{\sigma}}\right)^{1-\sigma}}\cdot\left[\bar{\alpha}(b_0)^{1-\sigma}-(\alpha)^{1-\sigma}\right],$$

or,

$$\begin{aligned}&[V^d_{I,1}(b_0)-V^r_{I,1}(b_0,\alpha)]+[V^r_{O,1}(b_0,\alpha)-V^d_{O,1}(b_0)]\\&\quad=\frac{1+\theta^{\frac{1}{\sigma}}-\theta-\theta^{\frac{1-\sigma}{\sigma}}}{1-\sigma}\cdot\left[\frac{\bar{\alpha}(b_0)(\tau-rb_0)}{(1+\theta^{\frac{1}{\sigma}})\,[1+(1+r)\Gamma(\delta)]}\right]^{1-\sigma}\\&\qquad\times\left\{[1+(1+r)\Gamma(\delta)]^{1-\sigma}-\left[(1+r)+\left(\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}}\right]^{1-\sigma}\cdot\Gamma(\delta)^{1-\sigma}\right\}.\end{aligned}\tag{D.4}$$

Finally, upon substituting equations (D.3) and (D.4) into equation (D.1), we derive the condition necessary for determining the optimal value of δ.

$$\begin{aligned}\delta&=\frac{\left[\begin{aligned}&\left(\theta+\theta^{\frac{1-\sigma}{\sigma}}\right)\cdot\left[(1+r)+\left(\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}}\right]^{1-\sigma}\cdot\Gamma(\delta)^{1-\sigma}\\&\quad+\beta\phi\cdot\left[(1+r)+\left(\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}}\right]^{1-\sigma}\\&\quad-[1+(1+r)\Gamma(\delta)]^{1-\sigma}\cdot\left[\theta+\theta^{\frac{1-\sigma}{\sigma}}+\beta(1+\theta^{\frac{1}{\sigma}})\left(\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}}\right]\end{aligned}\right]}{\left(1+\theta^{\frac{1}{\sigma}}-\theta-\theta^{\frac{1-\sigma}{\sigma}}\right)\cdot\left\{[1+(1+r)\Gamma(\delta)]^{1-\sigma}-\left[(1+r)+\left(\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}}\right]^{1-\sigma}\cdot\Gamma(\delta)^{1-\sigma}\right\}}\\&\equiv RHS(\delta).\end{aligned}\tag{D.5}$$

Then, the optimal supermajority threshold δ^{opt} is defined as the value of δ that satisfies $\delta=$ RHS(δ).

We establish the condition for the existence of a $\delta\in(1/2,1)$ that fulfills equation (D.5). First, we demonstrate that the denominator of RHS(δ) in (D.5) is positive. Using the property in (12), the requisite condition for the denominator of RHS(δ) to be positive is:

$$\begin{aligned}&[1+(1+r)\Gamma(\delta)]^{1-\sigma}-\left[(1+r)+\left(\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}}\right]^{1-\sigma}\cdot\Gamma(\delta)^{1-\sigma}>0\\&\Leftrightarrow\ (3-2\delta)(1+\theta^{\frac{1}{\sigma}})+(2\delta-1)(\theta+\theta^{\frac{1-\sigma}{\sigma}})>0,\end{aligned}$$

where the second inequality holds as $\delta\in[1/2,1]$. Consequently, the denominator in equation (D.5) is always positive.

If the following two conditions are satisfied, then there exists at least one $\delta \in (1/2, 1)$ that satisfies $\delta = \text{RHS}(\delta)$ in (D.5):

$$RHS\left(\frac{1}{2}\right) > \frac{1}{2}, \quad RHS(1) < 1.$$

To establish $RHS\left(\frac{1}{2}\right) > \frac{1}{2}$, we should note that $\Gamma\left(\frac{1}{2}\right) = 1$ (as indicated in equation (36)) and that $\beta(1+r) = 1$ (according to Assumption 2). Moreover, considering the definition of ϕ provided in equation (16), we obtain the following reformulation:

$$\begin{aligned} &RHS\left(\frac{1}{2}\right) > \frac{1}{2} \\ &\Leftrightarrow\ W(x) \equiv (2+r)^{\sigma} \cdot \left[(1+r) + \left(\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}}\right]^{1-\sigma} - \left[(1+r) + \left(\phi/\left(1+\theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}-1}\right] > 0. \end{aligned} \tag{D.6}$$

We define x as $x \equiv \phi/\left(1+\theta^{\frac{1}{\sigma}}\right)$ and denote the left-hand side of (D.6) as $W(x)$. Then the expression in (D.6) is further reformulated as follows:

$$RHS\left(\frac{1}{2}\right) > \frac{1}{2} \Leftrightarrow\ (2+r)^{\sigma} \cdot \left[(1+r) + (x)^{\frac{1}{\sigma}}\right]^{1-\sigma} - \left[(1+r) + (x)^{\frac{1}{\sigma}-1}\right] > 0.$$

The partial differentiation of $W(x)$ with respect to x yields:

$$\frac{\partial W(x)}{\partial x} = \frac{1-\sigma}{\sigma} \cdot x^{\frac{1}{\sigma}-2} \cdot \left\{\left[\frac{2+r}{(1+r) + x^{\frac{1}{\sigma}}}\right]^{\sigma} \cdot x - 1\right\},$$

where $\partial W(x)/\partial x < 0$ holds as long as the following condition holds:

$$\frac{\partial W(x)}{\partial x} < 0 \ \Leftrightarrow\ \left[\frac{2+r}{(1+r) + x^{\frac{1}{\sigma}}}\right]^{\sigma} \cdot x < 1 \Leftrightarrow\ x < 1. \tag{D.7}$$

From (12) and (16), we have

$$x \equiv \frac{\phi}{1+\theta^{\frac{1}{\sigma}}} = \frac{1+\theta^{\frac{1}{\sigma}} + \theta + \theta^{\frac{1-\sigma}{\sigma}}}{2\left(1+\theta^{\frac{1}{\sigma}}\right)} = \frac{1}{2} + \frac{\theta + \theta^{\frac{1-\sigma}{\sigma}}}{2\left(1+\theta^{\frac{1}{\sigma}}\right)} < \frac{1}{2} + \frac{1}{2} = 1,$$

showing that $\partial W(x)/\partial x < 0$ holds. Furthermore, given that $x < 1$, we obtain the following result:

$$W(x) > W(1) = (2+r)^{\sigma} \cdot [(1+r)+1]^{1-\sigma} - [(1+r)+1] = 0.$$

As $W(x) > 0$ consistently holds, (D.6) guarantees that $RHS\left(\frac{1}{2}\right) > \frac{1}{2}$ is always satisfied.

Next, we derive the condition for $RHS\,(1) < 1$ to be satisfied.

$RHS\,(1) < 1$

$$\Leftrightarrow \; 0 < \left[(1+r) + \left(\phi / \left(1+\theta^{\frac{1}{\sigma}}\right)\right)^{\frac{1}{\sigma}}\right]^{\sigma} \cdot [1+(1+r)\Gamma(1)]^{1-\sigma} - \left[(1+r)\Gamma(1)^{1-\sigma} + \frac{\phi}{1+\theta^{\frac{1}{\sigma}}}\right]. \tag{D.8}$$

We denote the right-hand side of (D.8) as $Z(x)$. Utilizing the definition of $\Gamma(\delta)$ provided in Lemma 2, we obtain the following.

$$\Gamma(1) = \left[1 + \frac{1+\theta^{\frac{1}{\sigma}} - \theta - \theta^{\frac{1-\sigma}{\sigma}}}{2\left(1+\theta^{\frac{1}{\sigma}} + \theta + \theta^{\frac{1-\sigma}{\sigma}}\right)}\right]^{\frac{1}{\sigma}} = \left(\frac{1}{2} + \frac{1}{2x}\right)^{\frac{1}{\sigma}}.$$

Using this expression, we can rewrite (D.8) as follows:

$$RHS\,(1) < 1 \;\Leftrightarrow\; Z(x) = \left[(1+r) + x^{\frac{1}{\sigma}}\right]^{\sigma} \cdot \left[1 + (1+r)\left(\frac{1}{2} + \frac{1}{2x}\right)^{\frac{1}{\sigma}}\right]^{1-\sigma}$$
$$- (1+r)\left(\frac{1}{2} + \frac{1}{2x}\right)^{\frac{1-\sigma}{\sigma}} - x > 0. \tag{D.9}$$

We now show that $Z(x) > 0$ holds for all $x \in [1/2, 1)$ using Hölder's inequality. Hölder's inequality states that for $p, q > 1$ with $1/p + 1/q = 1$ and non-negative reals a_i, b_i,

$$\sum_i a_i b_i \leq \left(\sum_i a_i^p\right)^{1/p} \left(\sum_i b_i^q\right)^{1/q},$$

with equality if and only if a_i^p / b_i^q is constant across all i.

Set $p = 1/\sigma > 1$ and $q = 1/(1-\sigma) > 1$, so that $1/p + 1/q = \sigma + (1-\sigma) = 1$. Define the sequences

$$(a_1,\; a_2) = \left((1+r)^{\sigma},\;\; x\right),$$
$$(b_1,\; b_2) = \left(\left[(1+r)\left(\frac{1}{2} + \frac{1}{2x}\right)^{\frac{1}{\sigma}}\right]^{1-\sigma},\;\; 1\right).$$

The left-hand side of Hölder's inequality is:

$$\sum_i a_i b_i = (1+r)^{\sigma} \cdot \left[(1+r)\left(\frac{1}{2} + \frac{1}{2x}\right)^{\frac{1}{\sigma}}\right]^{1-\sigma} + x$$
$$= (1+r)\left(\frac{1}{2} + \frac{1}{2x}\right)^{\frac{1-\sigma}{\sigma}} + x,$$

whereas the right-hand side of Hölder's inequality is:

$$\left(\sum_i a_i^{\frac{1}{\sigma}}\right)^{\sigma} \left(\sum_i b_i^{\frac{1}{1-\sigma}}\right)^{1-\sigma} = \left[(1+r) + x^{\frac{1}{\sigma}}\right]^{\sigma} \cdot \left[1 + (1+r)\left(\frac{1}{2} + \frac{1}{2x}\right)^{\frac{1}{\sigma}}\right]^{1-\sigma}.$$

Hölder's inequality thus gives:

$$(1+r)\left(\frac{1}{2}+\frac{1}{2x}\right)^{\frac{1-\sigma}{\sigma}}+x \leq \left[(1+r)+x^{\frac{1}{\sigma}}\right]^{\sigma} \cdot \left[1+(1+r)\left(\frac{1}{2}+\frac{1}{2x}\right)^{\frac{1}{\sigma}}\right]^{1-\sigma},$$

which implies $Z(x) \geq 0$.

The equality holds if and only if $a_1^{1/\sigma}/b_1^{1/(1-\sigma)} = a_2^{1/\sigma}/b_2^{1/(1-\sigma)}$, i.e.,

$$\frac{(1+r)}{(1+r)\left(\frac{1}{2}+\frac{1}{2x}\right)^{\frac{1}{\sigma}}} = \frac{x^{\frac{1}{\sigma}}}{1} \;\Leftrightarrow\; x=1.$$

Since $x \in [1/2, 1)$, the equality, $Z(x) = 0$, never holds; a strict inequality, $Z(x) > 0$, holds for $x \in (1/2, 1)$. This establishes $RHS(1) < 1$, which, together with $RHS(1/2) > 1/2$ shown above, guarantees the existence of at least one $\delta \in (1/2, 1)$ satisfying $\delta = RHS(\delta)$ in (D.5).

Lastly, as per Lemma 2, if $\delta^{opt} \in (1/2, 1)$ is satisfied, then it follows that $\alpha^{opt}(b_0) = \alpha^*(b_0, \delta^{opt}) \in (0, \bar{\alpha}(b_0))$ is also satisfied.

Q.E.D.

E Proof of Prpopsition 2

We initially establish that δ^{opt} is independent of b_0. The optimal supermajority threshold, denoted as δ^{opt}, is the value of δ that satisfies equation (D.5). Because equation (D.5) does not include b_0, δ^{opt} is independent of b_0.

We then demonstrate that α^{opt} is a decreasing function of b_0. Recall that δ^{opt} is independent of b_0. Moreover, considering the expression in (35), we have:

$$\begin{aligned}\frac{\partial \alpha^{opt}}{\partial b_0} &= \frac{\Gamma(\delta^{opt})}{1+(1+r)\Gamma(\delta^{opt})} \cdot \frac{\partial}{\partial b_0}\left(\frac{\tau+(1+r)\left[\tau-(1+r)b_0\right]}{\tau - r b_0}\right) \\ &= \frac{\Gamma(\delta^{opt})}{1+(1+r)\Gamma(\delta^{opt})} \cdot \frac{-\tau}{(\tau - r b_0)^2} \\ &< 0,\end{aligned}$$

showing that α^{opt} is a decreasing function of b_0.

Q.E.D.