X_i is a random variable.

Consider the distribution of

$$\frac{1}{n}\frac{\partial \log L(\theta)}{\partial\theta} = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial \log f(X_i;\theta)}{\partial\theta}.$$

We have to obtain mean and variance of $\dfrac{\partial \log f(X_i;\theta)}{\partial\theta}$.

Suppose that X_i is a continuous type of random variable.

$f(x_i;\theta)$ denotes the density function.

Therefore, we have:

$$\int f(x_i;\theta)\mathrm{d}x_i = 1$$

Taking the derivative with respect to θ on both sides, we obtain:

$$0 = \int \frac{\partial f(x_i;\theta)}{\partial\theta}\mathrm{d}x_i = \int \frac{\partial \log f(x_i;\theta)}{\partial\theta}f(x_i;\theta)\mathrm{d}x_i = \mathrm{E}\Big(\frac{\partial \log f(X_i;\theta)}{\partial\theta}\Big)$$

Again, take the derivative with respect to θ on both sides as follows:

$$
\begin{aligned}
0 &= \int \frac{\partial^2 \log f(x_i;\theta)}{\partial\theta\partial\theta'} f(x_i;\theta) + \frac{\partial \log f(x_i;\theta)}{\partial\theta}\frac{\partial f(x_i;\theta)}{\partial\theta'} \mathrm{d}x_i \\
&= \int \frac{\partial^2 \log f(x_i;\theta)}{\partial\theta\partial\theta'} f(x_i;\theta)\mathrm{d}x_i + \int \frac{\partial \log f(x_i;\theta)}{\partial\theta}\frac{\partial \log f(x_i;\theta)}{\partial\theta'} f(x_i;\theta)\mathrm{d}x_i \\
&= \mathrm{E}\Big(\frac{\partial^2 \log f(X_i;\theta)'}{\partial\theta\partial\theta}\Big) + \mathrm{E}\Big(\frac{\partial \log f(X_i;\theta)}{\partial\theta}\frac{\partial \log f(X_i;\theta)}{\partial\theta'}\Big),
\end{aligned}
$$

i.e.,

$$
-\mathrm{E}\Big(\frac{\partial^2 \log f(X_i;\theta)'}{\partial\theta\partial\theta}\Big) = \mathrm{E}\Big(\frac{\partial \log f(X_i;\theta)}{\partial\theta}\frac{\partial \log f(X_i;\theta)}{\partial\theta'}\Big) = \mathrm{V}\Big(\frac{\partial \log f(X_i;\theta)}{\partial\theta}\Big) = \Sigma_i
$$

Thus, $\dfrac{\partial \log f(X_i;\theta)}{\partial\theta}$ is distributed with mean 0 and variance Σ_i.

Note as follows:

$$
I(\theta) = -\mathrm{E}\Big(\frac{\partial^2 \log L(\theta)}{\partial\theta\partial\theta'}\Big) = -\sum_{i=1}^{n} \mathrm{E}\Big(\frac{\partial^2 \log f(X_i;\theta)}{\partial\theta\partial\theta'}\Big) = \sum_{i=1}^{n} \Sigma_i.
$$

Using the central limit theorem (generalization) shown above, asymptotically we obtain the following distribution:

$$\frac{1}{\sqrt{n}}\frac{\partial \log L(\theta)}{\partial\theta} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\partial \log f(X_i;\theta)}{\partial\theta} \longrightarrow N(0,\Sigma),$$

where $\Sigma = \lim_{n\to\infty}\Big(\frac{1}{n}I(\theta)\Big)$.

Let $\tilde{\theta}$ be the maximum likelihood estimator.

Note that the MLE $\tilde{\theta}$ satisfies:

$$\frac{\partial \log L(\tilde{\theta})}{\partial\theta} = \sum_{i=1}^{n}\frac{\partial \log f(X_i;\tilde{\theta})}{\partial\theta} = 0.$$

Linearizing $\dfrac{\partial \log L(\tilde{\theta})}{\partial\theta}$ around $\tilde{\theta} = \theta$, we obtain:

$$0 = \frac{1}{\sqrt{n}}\frac{\partial \log L(\tilde{\theta})}{\partial\theta} \approx \frac{1}{\sqrt{n}}\frac{\partial \log L(\theta)}{\partial\theta} + \frac{1}{\sqrt{n}}\frac{\partial^2 \log L(\theta)}{\partial\theta\partial\theta'}(\tilde{\theta}-\theta),$$

where the rest of terms (i.e., the second-order term, the third-order term, ...) are ignored, which implies that the distribution of $\dfrac{1}{\sqrt{n}}\dfrac{\partial \log L(\theta)}{\partial \theta}$ is asymptotically equivalent to that of $\dfrac{1}{\sqrt{n}}\dfrac{\partial^2 \log L(\theta)}{\partial \theta \partial \theta'}(\tilde{\theta} - \theta)$.

We have already known the distribution of $\dfrac{1}{\sqrt{n}}\dfrac{\partial \log L(\theta)}{\partial \theta}$ as follows:

$$\frac{1}{\sqrt{n}}\frac{\partial \log L(\theta)}{\partial \theta} \approx -\frac{1}{\sqrt{n}}\frac{\partial^2 \log L(\theta)}{\partial \theta \partial \theta'}(\tilde{\theta} - \theta) = \left(-\frac{1}{n}\frac{\partial^2 \log L(\theta)}{\partial \theta \partial \theta'}\right)\sqrt{n}(\tilde{\theta} - \theta) \longrightarrow N(0, \Sigma).$$

Note as follows:

$$-\frac{1}{n}\frac{\partial^2 \log L(\theta)}{\partial \theta \partial \theta'} \longrightarrow \lim_{n\to\infty}\left(\frac{1}{n}\mathrm{E}\Big(-\frac{\partial^2 \log L(\theta)}{\partial \theta \partial \theta'}\Big)\right) = \lim_{n\to\infty}\Big(\frac{1}{n}I(\theta)\Big) = \Sigma.$$

Thus, $\left(-\dfrac{1}{n}\dfrac{\partial^2 \log L(\theta)}{\partial \theta \partial \theta'}\right)\sqrt{n}(\tilde{\theta}-\theta)$ asymptotically has the same distribution as $\Sigma\sqrt{n}(\tilde{\theta}-\theta)$.

Therefore,

$$\mathrm{V}(\Sigma\sqrt{n}(\tilde{\theta} - \theta)) = \Sigma \mathrm{V}(\sqrt{n}(\tilde{\theta} - \theta))\Sigma' \longrightarrow \Sigma.$$

Note that $\Sigma = \Sigma'$. Thus, we have the asymptotic variance of $\sqrt{n}(\tilde{\theta} - \theta)$ as follows:

$$\mathrm{V}(\sqrt{n}(\tilde{\theta} - \theta)) \longrightarrow \Sigma^{-1}\Sigma\Sigma^{-1} = \Sigma^{-1}.$$

Finally, we obtain:

$$\sqrt{n}(\tilde{\theta} - \theta) \longrightarrow N(0, \Sigma^{-1}).$$

11 Consistency and Asymptotic Normality of OLSE

Regression model: $y = X\beta + u, \qquad u \sim (0, \sigma^2 I_n).$

Consistency:

1. Let $\hat{\beta}_n = (X'X)^{-1}X'y$ be the OLS with sample size n.

   Consistency: As n is large, $\hat{\beta}_n$ converges to β.

2. Assume the stationarity condition for X, i.e.,

$$\frac{1}{n}X'X \longrightarrow M_{xx}.$$

   and no correlation between X and u, i.e.,

$$\frac{1}{n}X'u \longrightarrow 0.$$

3. Note that $\frac{1}{n}X'X \longrightarrow M_{xx}$ results in $(\frac{1}{n}X'X)^{-1} \longrightarrow M_{xx}^{-1}$.

4. OLS is given by:

$$\hat{\beta}_n = \beta + (X'X)^{-1}X'u = \beta + (\frac{1}{n}X'X)^{-1}(\frac{1}{n}X'u).$$

Therefore,

$$\hat{\beta}_n \longrightarrow \beta + M_{xx}^{-1} \times 0 = \beta$$

Thus, OLSE is a consistent estimator.

Asymptotic Normality:

1. Asymptotic Normality of OLSE

$$\sqrt{n}(\hat{\beta}_n - \beta) \longrightarrow N(0.\sigma^2 M_{xx}^{-1}), \quad \text{when } n \longrightarrow \infty.$$

2. **Central Limit Theorem:** Greenberg and Webster (1983)

Z_1, Z_2, $\cdots$, Z_n are mutually independent. Z_i is distributed with mean μ and variance Σ_i for $i = 1, 2, \cdots, n$.

Then, we have the following result:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Z_i - \mu) \longrightarrow N(0, \Sigma),$$

where

$$\Sigma = \lim_{n \to \infty} \left(\frac{1}{n} \sum_{i=1}^{n} \Sigma_i \right).$$

Note that the distribution of Z_i is not assumed.

3. Define $Z_i = x_i' u_i$. Then, $\Sigma_i = \mathrm{V}(Z_i) = \sigma^2 x_i' x_i$.

4. Σ is defined as:

$$\Sigma = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\sigma^2 x_i' x_i\right) = \sigma^2 \lim_{n\to\infty}\left(\frac{1}{n}X'X\right) = \sigma^2 M_{xx},$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

5. Applying Central Limit Theorem (Greenberg and Webster (1983), we obtain the following:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i' u_i = \frac{1}{\sqrt{n}}X'u \longrightarrow N(0, \sigma^2 M_{xx}).$$

On the other hand, from $\hat{\beta}_n = \beta + (X'X)^{-1}X'u$, we can rewrite as:

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{1}{n}X'X\right)^{-1}\frac{1}{\sqrt{n}}X'u.$$

$$
\begin{aligned}
\mathrm{V}\Big(\big(\frac{1}{n}X'X\big)^{-1}\frac{1}{\sqrt{n}}X'u\Big) &= \mathrm{E}\Big(\big(\frac{1}{n}X'X\big)^{-1}\frac{1}{\sqrt{n}}X'u\big(\big(\frac{1}{n}X'X\big)^{-1}\frac{1}{\sqrt{n}}X'u\big)'\Big) \\
&= \big(\frac{1}{n}X'X\big)^{-1}\big(\frac{1}{n}X'\mathrm{E}(uu')X\big)\big(\frac{1}{n}X'X\big)^{-1} \\
&= \sigma^2\big(\frac{1}{n}X'X\big)^{-1}\big(\frac{1}{n}X'X\big)\big(\frac{1}{n}X'X\big)^{-1} \\
&\longrightarrow \ \sigma^2 M_{xx}^{-1} M_{xx} M_{xx}^{-1} = \sigma^2 M_{xx}^{-1}.
\end{aligned}
$$

Therefore,

$$
\sqrt{n}(\hat{\beta} - \beta) \ \longrightarrow \ N(0, \sigma^2 M_{xx}^{-1})
$$

$\Longrightarrow$ Asymptotic normality (漸近的正規性) of OLSE

The distribution of u_i is not assumed.

12 Instrumental Variable (操作変数法)

12.1 Measurement Error (測定誤差)

Errors in Variables

1. True regression model:

$$y = \tilde{X}\beta + u$$

2. Observed variable:

$$X = \tilde{X} + V$$

   V: is called the **measurement error (測定誤差 or 観測誤差)**.

3. For the elements which do not include measurement errors in X, the corresponding elements in V are zeros.

4. Regression using observed variable:

$$y = X\beta + (u - V\beta)$$

OLS of β is:

$$\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'(u - V\beta)$$

5. Assumptions:

   (a) The measurement error in X is uncorrelated with $\tilde{X}$ in the limit. i.e.,

$$\text{plim}\Big(\frac{1}{n}\tilde{X}'V\Big) = 0.$$

   Therefore, we obtain the following:

$$\text{plim}\Big(\frac{1}{n}X'X\Big) = \text{plim}\Big(\frac{1}{n}\tilde{X}'\tilde{X}\Big) + \text{plim}\Big(\frac{1}{n}V'V\Big) = \Sigma + \Omega$$