2025/06/24

Econometrics I's Homework #2

Deadline: July 22, 2025, 10:20AM

- The answer should be written in English or Japanese.
- Your name and student ID number should be included in your answer sheet.

1 Consider the following regression model:

$$y = X\beta + u = i\beta_1 + X_2\beta_2 + u$$

where y, $X = (i, X_2)$, $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ and u denote $T \times 1$, $T \times k$, $k \times 1$ and $T \times 1$ matrices. k and T are the numberof explanatory variables and the sample size. Note that $i = (1, 1, \cdots, 1)'$. β_1 is a constant term. β_2 is a $(k-1) \times 1$ vector, and X_2 is a $T \times (k-1)$ matrix. u_1, u_2, $\cdots$, u_T are mutually independently and **normally** distributed with mean zero and variance σ^2, i.e., $u \sim N(0, \sigma^2 I_T)$. β are a vector of unknown parameters to be estimated. Let $\hat{\beta}$ be the ordinary least squares estimator of β. We define a vector of residuals as $e = y - X\hat{\beta}$.

(1) Show that $Mi = 0$ and $Me = e$, where $M = I_T - \frac{1}{T}ii'$.

(2) Show that $e'e = y'My - \hat{\beta}'X'MX\hat{\beta} = y'My - \hat{\beta}_2'X_2'MX_2\hat{\beta}_2$, where $\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}$.

(3) The coefficient of determination is defined as: $R^2 = 1 - \dfrac{e'e}{y'My}$, where $M = I_T - \dfrac{1}{T}ii'$.

Show that $R^2 = \dfrac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2}{y'My}$

(4) Let R be an $G \times k$ vector with $G \leq k$. Derive the distribution of $R\hat{\beta}$.

(5) The null hypothesis is given by $H_0: \ R\beta = r$.

Under the null hypothesis, derive the distribution of $R\hat{\beta}$.

(6) We want to test $\beta_2 = 0$. What are G, R and r?

(7) We know that $\dfrac{(R\hat{\beta}-r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta}-r)/G}{e'e/(T-k)} \sim F(G, T-k)$.

Show that $\dfrac{\hat{\beta}_2' X_2' M X_2 \hat{\beta}_2/G}{e'e/(T-k)} \sim F(G, T-k)$.

(8) When we test $H_0:\ \beta_2 = 0$, the test statistic is given by $\dfrac{R^2/(k-1)}{(1-R^2)/(T-k)}$.

Show that $\dfrac{R^2/(k-1)}{(1-R^2)/(T-k)} \sim F(k-1, T-k)$.

(9) Suppose that we have the following estimation result:

$$y_i = \underset{(3.163)}{0.3} + \underset{(0.240)}{0.65}\ X_i, \qquad s = 1.07, \quad R^2 = 0.786$$

where the sample size is $T = 4$, ( ) indicates the standard error of the corresponding coefficient estimate, s denote the standard error of regression, and R^2 represents the coefficient of determination.

Let β be the coefficient of X_i.

We want to test $\beta = 0$ for the regression model: $y_i = \alpha + \beta X_i + u_i$.

First, using the estimate 0.65 and its standard error of coefficient (i.e., 0.240), test $\beta = 0$.

Second, using R^2, test $\beta = 0$.

Compare both test statistics and make sure $t^2(T-k) = F(1, T-k)$

You can find various distribution tables in standard introductory statistics textbooks.

2 Consider the regression model:

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2 I_T),$$

where y, X, β and u are $T\times 1$, $T\times k$, $k\times 1$ and $T\times 1$.

Let $\hat{\beta}$ be the ordinary least squares estimator, and $\tilde{\beta}$ be the ordinary least squares estimator restricted to $R\beta = r$, where R and r are $G\times k$, $G\times 1$ and $G \le k$. $\hat{u}$ and $\tilde{u}$ are defined as the OLS residual and the restricted OLS residual, respectively.

$$y = X\hat{\beta} + \hat{u} \tag{A}$$

$$y = X\tilde{\beta} + \tilde{u}, \qquad R\tilde{\beta} = r \tag{B}$$

(1) Derive the restricted OLS $\tilde{\beta}$.

(2) Show the following:

$$\frac{(\tilde{u}'\tilde{u} - \hat{u}'\hat{u})/G}{\hat{u}'\hat{u}/(T-k)} \sim F(G, T-k).$$

(3) Suppose that the coefficients of determination in (A) and (B) are given $\hat{R}^2$ and $\tilde{R}^2$, respectively. Show the following:

$$\frac{(\hat{R}^2 - \tilde{R}^2)/G}{(1-\hat{R}^2)/(T-k)} \sim F(G, T-k).$$

3 We consider estimating the following three production functions.

$$\log(Y_t) = \alpha_0 + \alpha_1 \log(K_t) + \alpha_2 \log(L_t) + u_t$$

$$\log(y_t) = \beta_0 + \beta_1 \log(k_t) + u_t$$

$$\log(Y_t) = \gamma_0 + \gamma_1 \log(K_t) + \gamma_2 \log(L_t) + \gamma_3 D_t + \gamma_4 D_t \log(K_t) + \gamma_5 D_t \log(L_t) + u_t$$

The estimation period is 1969 – 1997 (it's too old!). Let Y_t be GDP (10 billion yen, 1992 price), K_t be the net worth (10 billion yen, deflated by the GDP deflator), L_t be the number of employees, D_t be the dummy variable, which is one after 1991 and zero before 1991, y_t be the per capita GDP (10 billion yen, 1992 price, $y_t = Y_t/L_t$), and k_t be the per capita net worth (10 billion yen, deflated by the GDP deflator, $k_t = K_t/L_t$). The error terms $u_1, u_2, \cdots, u_T$ are mutually independently, identically and normally distributed.

The following estimation results are obtained.

$$\log(Y_t) = \underset{(7.283)}{-\ 30.6242} + \underset{(5.054)}{.230042} \log(K_t) + \underset{(8.266)}{2.23565} \log(L_t)$$

$$R^2 = .986684, \quad \overline{R}^2 = .985659, \quad \widehat{\sigma}^2 = .00141869$$

$$\log(y_t) = \underset{(41.08)}{-\ 3.53058} + \underset{(19.62)}{.504043} \log(k_t)$$

$$R^2 = .934448, \quad \overline{R}^2 = .932020, \quad \widehat{\sigma}^2 = .00354801$$

$$\log(Y_t) = \underset{(3.630)}{-\ 34.6168} + \underset{(2.588)}{.204302} \log(K_t) + \underset{(4.155)}{2.48045} \log(L_t)$$

$$-\ \underset{(1.090)}{54.8287}\ D_t + \underset{(.4665)}{.243766}\ D_t \log(K_t) + \underset{(1.134)}{2.84275}\ D_t \log(L_t)$$

$$R^2 = .987960, \quad \overline{R}^2 = .985342, \quad \widehat{\sigma}^2 = .00145010$$

Note that the values in the parentheses denote the t values, R^2 is the coefficient of determination, $\overline{R}^2$ is the adjusted R^2, and $\widehat{\sigma}^2$ is the variance estimate of regression.

Answer the following questions.

(1) Test $H_0: \ \alpha_1 = \alpha_2 = 0$.

(2) Test whether the production function is homogeneous.

(3) Test whether the structural change occurred after 1991.

For each question, show the testing procedure in detail.

4

(1) Show that there exists P such that $\Omega = PP'$ when Ω is a symmetric and positive definite matrix.

(2) Consider the following regression model:

$$y_t = x_t\beta + u_t, \qquad u_t \sim N(0, \sigma^2 z_t^2), \qquad t = 1, 2, \cdots, T,$$

where u_1, u_2, $\cdots$, u_T are mutually independent.

What is the variance-covariance matrix, denoted by $\sigma^2\Omega$, of $u = (u_1, u_2, \cdots, u_T)'$.

(3) Consider the following regression model:

$$y_t = x_t\beta + u_t, \qquad u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim N(0, \sigma^2), \qquad t = 1, 2, \cdots, T,$$

where ϵ_1, ϵ_2, $\cdots$, ϵ_T are mutually independent.

What is the variance-covariance matrix, denoted by $\sigma^2\Omega$, of $u = (u_1, u_2, \cdots, u_T)'$.

(4) Let b be the best linear unbiased estimator of β under the following regression model:

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2\Omega),$$

where y, X, β and u are $T \times 1$, $T \times k$, $k \times 1$ and $T \times 1$.

Derive b.

(5) Consider the regression model in (4). We have two estimators, $\hat{\beta}$ and b, to estimate β, where $\hat{\beta}$ denotes the OLS estimator and b indicates the GLS estimator.

Obtain $\mathrm{E}(\hat{\beta})$ and $\mathrm{V}(\hat{\beta})$.

Show that $\mathrm{V}(\hat{\beta}) - \mathrm{V}(b)$ is a positive definite matrix.

5 Suppose that u_1, u_2, $\cdots$, u_T are mutually independently and normally distributed with $\mathrm{E}(u_t) = 0$ and $\mathrm{V}(u_t) = \sigma^2$ for all $t = 1, 2, \cdots, T$.

Consider the following regression model:

$$y_t = \alpha + \beta x_t + u_t$$

Answer the following questions.

(1) Construct the likelihood function of $(\alpha, \beta, \sigma^2)$.

(2) Obtain the maximum likelihood estimator of $\theta = (\alpha, \beta, \sigma^2)'$, denoted by $\tilde{\theta} = (\tilde{\alpha}, \tilde{\beta}, \tilde{\sigma}^2)'$.

(3) Obtain the information matrix $I(\theta)$.

(4) Compare OLSE (i.e., $\hat{\beta}$) and MLE (i.e., $\tilde{\beta}$) of β with respect to mean and variance. How about OLSE (i.e., s^2) and MLE (i.e., $\hat{\sigma}^2$) of σ^2?

6 Suppose that ϵ_1, ϵ_2, $\cdots$, ϵ_T are mutually independently and normally distributed with $\mathrm{E}(\epsilon_t) = 0$ and $\mathrm{V}(\epsilon_t) = \sigma^2$ for all $t = 1, 2, \cdots, T$.

Consider the following regression model:

$$y_t = \rho y_{t-1} + \epsilon_t, \qquad \epsilon_t \sim N(0, \sigma^2)$$

Answer the following questions.

(1) Obtain mean and variance of y for $y = (y_1, y_2, \cdots, y_T)'$.

(2) Using (1), construct the likelihood function of (ρ, σ^2).

(3) Obtain unconditional mean and variance of y_t. Obtain conditional mean and variance of y_t given y_{t-1}, y_{t-2}, $\cdots$.

(4) Using (3), construct the likelihood function in the innovation form.

(5) Comparing the two likelihood function, obtain the matrix P such that $\Omega = PP'$, where Ω denotes the variance obtained in (2).

7 Suppose that u_1, u_2, $\cdots$, u_T are mutually independently distributed with $\mathrm{E}(u_t) = 0$ and $\mathrm{V}(u_t) = \sigma^2$ for all $t = 1, 2, \cdots, T$.

Consider the following regression model:

$$y = X\beta + u,$$

where y, X, β and u are $T \times 1$, $T \times k$, $k \times 1$ and $T \times 1$ matrices or vectors. Answer the following questions.

(1) Let $\hat{\beta}$ be the ordinary least squares estimator of β. Show that $\hat{\beta}$ is a consistent estimator of β. You have to make clear the underlying assumptions.

(2) As T goes to infinity, what is the asymptotic distribution of $\dfrac{1}{\sqrt{T}}X'u$?

(3) Obtain the asymptotic distribution of $\sqrt{T}(\hat{\beta}-\beta)$.

8 Suppose that X_1, X_2, $\cdots$, X_T are mutually independently distributed with the density functions $f(x_i;\theta)$, $i=1,2,\cdots,T$.

(1) Let $\hat{\theta}$ be the maximun likelihood estimator of θ. Show that $\hat{\theta}$ is a consistent estimator of θ. You have to make clear the underlying assumptions.

(2) As T goes to infinity, what is the asymptotic distribution of $\dfrac{1}{\sqrt{T}}\sum_{i=1}^{T}\dfrac{\partial \log f(X_i;\theta)}{\partial\theta}$?

(3) Obtain the asymptotic distribution of $\sqrt{T}(\hat{\theta}-\theta)$.

9 Suppose that u_1, u_2, $\cdots$, u_T are mutually independently distributed with $\mathrm{E}(u_t)=0$ and $\mathrm{V}(u_t)=\sigma^2$ for all $t=1,2,\cdots,T$.

Consider the following regression model:

$$y = X\beta + u,$$

where y, X, β and u are $T\times 1$, $T\times k$, $k\times 1$ and $T\times 1$ matrices or vectors. Answer the following questions.

(1) When X is correlated with u, show that OLSE of β, i.e., $\hat{\beta}$, is inconsistent.

(2) X is correlated with u. Suppose that we have the $T\times k$ matrix Z which is uncorrelated with u and correlated with X. Obtain a consistent estimator of β, usin Z.

(3) Obtain an asymptotic distribution of the consistent estimator in (2).